Thesis

July 25, 2026

NOTE: Massive notation inconsistency between sections!!

Contents

1 Introducing new discrete Grisanov Transforms

The Basics of Polymer Models The first mathematically rigorous definition of a *directed polymer model* was developed by Imbrie and Spencer [IS88]. The model consists of

- $S = \{S_n\}_{n\geq 0}$ which is a simple random walk (SRW) on the lattice $\mathbb{Z}^d$. The random sequence is defined on the probability space $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{F}, P_x^{\mathrm{SRW}})$, where $\mathcal{F}$ is the *cylindrical* sigma algebra and P_x^{SRW} is a path measure such that,

$$P_x^{\mathrm{SRW}}(S_0 = x) = 1 \qquad P_x^{\mathrm{SRW}}(S_n - S_{n-1} = \pm e_j) = 1/2d \qquad j = 1, 2 \dots d \tag{1}$$

  where, e_j is the j^{th} vector for the canonical basis for $\mathbb{Z}^d$. We will use the notation $P_x^{\mathrm{SRW}}[X]$ to denote the expectation of a random variable X with respect to the measure P_X^{SRW}. When $x = 0$, we drop the subscript and use the notation $P_0 := P$. We will use the notation $\mathbf{x}$ to denote a realization of the SRW.

- A random environment $\omega = \{\omega(n, x) : n \in \mathbb{N}, x \in \mathbb{Z}^d\}$, which is a sequence of i.i.d, non-constant and real valued random variables defined on a probability space $(\Omega, \mathcal{W}, Q)$. We assume that $\omega(n, x)$ has moments of all order by declaring that

$$\lambda(\beta) := log(Q[e^{\beta\omega(n,x)}]) < \infty. \tag{2}$$

  We again adopt the notation that $Q[Y]$ to denote the expectation of a random variable Y with respect to the measure Q.

- The random path measure $P_n^{\beta,\omega}$, that is defined on the measurable space $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{F})$ through it's Radon-Nikodym derivative with the path measure P,

$$P_n^{\beta,\omega}(S) = \frac{e^{\beta H_n(\mathbf{x})}}{Z_n(\omega, \beta)} P^{\mathrm{SRW}}(S) \tag{3}$$

We call the path measure $P_n^{\beta,\omega}$ *polymer measure* and the simple random walk equipped with this measure, we call a *polymer*. The constant $\beta > 0$ is called the *inverse temperature* and

$$H_n(\mathbf{x}) := H_n^{\omega}(\mathbf{x}) = \sum_{1 \leq i \leq n} \omega(i, x_i) \tag{4}$$

is the total energy of a realized path $\mathbf{x}$ called the *Hamiltonian* potential. $Z_n(\omega, \beta)$ is a normalizing constant called the *partition function* that makes $P_n^{\beta,\omega}$ into a probability measure

$$Z_n(\omega, \beta) := Z_n = P[e^{\beta H_n(\mathbf{x})}]. \tag{5}$$

When $\beta = 0$, the polymer measure is just a simple random walk. One main object of interest is the *normalized* partition function

$$M_n = e^{-n\lambda(\beta)} Z_n. \tag{6}$$

One can check that (6) is a non-negative martingale with respect to the filtration $\mathcal{W}_n = \sigma\{w(i, x) : i \leq n, x \in \mathbb{Z}^d\}$; see [Com17]. Therefore, one can apply *Doob's Martingale Convergence Theorem* to M_n and consider its limit,

$$\lim_{n \to \infty} M_n = M_\infty \text{ a.s. } Q. \tag{7}$$

and by Komologrov's 0-1 law, the limit M_∞ is either strictly positive or identically zero almost surely Q. This provides the following dichotomy.

Definition 1. *We have the following classifications:*

- *If $Q(W_\infty = 0) = 1$ (a.s.), then we say that the polymer is in* **strong disorder**.
- *If $Q(W_\infty > 0) = 1$ (a.s.), then we say that the polymer is in* **weak disorder**.

Comets and Yoshida [CY06] showed that in all dimensions there is exists a critical temperature β_c such that for all $\beta < \beta_c$ the polymer is in weak disorder and for $\beta > \beta_c$ the polymer is in strong disorder. Further they showed that for $d = 1, 2$, $\beta_c = 0$. In [AKQ14b] they showed that with $d = 1$ and scaling the inverse temperature $\beta_n := n^{-1/4}$ the normalized partition and diffsuve scaling in space and time, the partition function no longer converges to zero, but rather has a limiting distribution which can be identified as a Wiener Chaos expansion. This intern can be identified as the *chaos* solution of Parabolic Anderson Model, see [AKQ14a] and [BC95]. **NOTE:** Should I put that result here?. The critical scaling of β_n is critical for the distributional convergence and characterizes and new regime for polymer model called intermediate disorder. In this paper we are interested in how certain changes of the SRW measure effects the polymer in intermediate disorder. For fixed $h \in \mathbb{R}$, we define a new path measure P_h^{TILT} on the same probability space as P^{SRW}.

The Tilted Measure We define the tilted measure by

$$P_h^{\mathrm{TILT}}(S) = \prod_{i=1}^{n} \frac{e^{h(S_i - S_{i-1})}}{\gamma(h)} P^{\mathrm{SRW}}(S) = \frac{e^{hS_n}}{\gamma(h)^n} P^{\mathrm{SRW}}(S) \tag{8}$$

where

$$\gamma(h) = P^{\mathrm{SRW}}\left[e^{h(S_i - S_{i-1})}\right] = \cosh(h). \tag{9}$$

Note that $P_0^{\mathrm{TILT}} = P^{\mathrm{SRW}}$ and that the measure P_h^{TILT} inherits the Markov property from P^{SRW}. Let $\mathcal{F}_n = \sigma(S_0, S_1, \ldots, S_n)$ be the sigma-algebra generated by the random walk up to time n. For $x, y \in \mathbb{Z}$,

$$P_h^{\mathrm{TILT}}(S_n = x \mid S_{n-1} = y) = \frac{e^{h(x-y)}}{\gamma(h)} P^{\mathrm{SRW}}(S_n = x \mid S_{n-1} = y). \tag{10}$$

Since the tilted measure factors as

$$\frac{e^{hS_n}}{\gamma(h)^n} = \frac{e^{hS_{n-1}}}{\gamma(h)^{n-1}} \cdot \frac{e^{h(S_n - S_{n-1})}}{\gamma(h)} \tag{11}$$

the factor $\frac{e^{hS_{n-1}}}{\gamma(h)^{n-1}}$ is $\mathcal{F}_{n-1}$-measurable leaving a transition kernel that depends only on S_{n-1}. Thus $P_h^{\mathrm{TILT}}(S_n = x \mid \mathcal{F}_{n-1})$ depends only on S_{n-1}, so the walk remains Markov under P_h^{TILT}. We use the notation E_h^{TILT} to denote taking expectation with respect to the measure P_h^{TILT}. We wish to consider the effect that this change of measure has on the Hamiltonian of the polymer model.

$$E_h^{\mathrm{TILT}}[\exp(\beta H_n(S))]\,. \tag{12}$$

We can calculate the partition function under this change of measure:

$$Z_n^{\mathrm{TILT}} := E_h^{\mathrm{TILT}}[\exp(\beta H_n(S))] \tag{13}$$

We can also write down the *tilted* point-to-point partition function

$$Z_n^{\mathrm{TILT}}(x) := E_h^{\mathrm{TILT}}\left[\exp(\beta H_n(S))\, \mathbf{1}_{\{S_n = x\}}\right] \tag{14}$$

and also the conditioned point-point partition function,

$$Z_n^{TILT}(x, t; y, s) = E_h^{\mathrm{TILT}}\left[\exp(\beta H_n(S))\, \mathbf{1}_{\{S_n = x\}} | S_n = y\right] \tag{15}$$

We define the polymer end-point measure as

$$P_{n,\beta_n}^{\omega, TILT}(S_n = x) = \frac{Z_n^{\mathrm{TILT}}(x)}{Z_n^{\mathrm{TILT}}}. \tag{16}$$

The Measure P_h^{AREA} We define the area measure by

$$P_h^{\mathrm{AREA}}(S) = \prod_{i=1}^{n} \frac{e^{h(S_i)}}{\gamma(h)} P_h^{\mathrm{TILT}}(S) = \frac{e^{h\sum_{i=1}^{n} S_n}}{\kappa(h)^n} P_h^{\mathrm{TILT}}(S) \tag{17}$$

where

$$\kappa(h) = P^{\mathrm{SRW}}\left[e^{h(S_i)}\right] = \cosh(kh).$$

2 Main Results

Recall the definition of the tilted partition function Z_n^{TILT} given in (13).

Theorem 2.1. *Assume that the random environment has mean zero and variance one. Under the scaling of $\beta_n = \beta n^{-1/4}$ and $h_n = hn^{-1/4}$, as $n \to \infty$, we have*

$$e^{-n\lambda(\beta_n)} Z_n^{TILT} \xrightarrow{d} \mathcal{Z}_{\sqrt{2}\beta,h} \tag{18}$$

where $\mathcal{Z}_{\sqrt{2}\beta,h}$ can be identified as an infinite sum of stochastic integrals

$$\mathcal{Z}_{\sqrt{2}\beta,h} = 1 + \sum_{k=1}^{\infty} (\sqrt{2}\beta)^k \int_{\Delta_k} \int_{\mathbb{R}^k} \prod_{i=1}^{k} W(t_i, x_i) \rho_h^{TILT}(t_i - t_{i-1}, x_i - x_{i-1})\, dx_i\, dt_i \tag{19}$$

where $W(t,x)$ is white noise on $\mathbb{R}^+ \times \mathbb{R}$, with covariance $E[W(t,x)W(s,y)] = \delta(t-s)\delta(x-y)$, $x_i \in \mathbb{R}$, $\Delta_k = \{0 \le t_0 < t_1 < \cdots < t_k \le 1\}$, and

$$\rho_h^{TILT}(t,x) = \frac{2}{\sqrt{2\pi t}} \exp\left(-\frac{(x-ht)^2}{2t}\right). \tag{20}$$

See Theorem A.1 for a full account of the local limit theorem in this case.

Theorem 2.2. *Consider the four-parameter field*

$$(x,t;s,y) \mapsto e^{-n(t-s)\lambda(\beta_n)} Z_n^{TILT}(nt, \sqrt{n}x, ns, \sqrt{n}y). \tag{21}$$

This four-parameter field converges weakly on the space of continuous bounded functions to the four-parameter process $\mathcal{Z}_{\beta,h}(x,t;s,y)$, which solves the following stochastic partial differential equation:

$$\partial_t \mathcal{Z}_{\beta,h} - \frac{1}{2}\partial_x^2 \mathcal{Z}_{\beta,h} + h\partial_x \mathcal{Z}_{\beta,h} - \beta W \mathcal{Z}_{\beta,h} = 0, \tag{22}$$

with initial condition

$$\lim_{t \downarrow s} \mathcal{Z}_{\beta,h}(s,y;t,x) = \delta_0(x-y). \tag{23}$$

Relationship with Polynomial Chaos Expansions At first glance, our model appears similar to the model constructed in [CSZ17]. However, the underlying models are fundamentally different. We briefly recall their construction and explain the distinction.

Let $\Omega_\delta = ((\delta\mathbb{Z} \times \delta^{1/2}\mathbb{Z}^d) \cap \Omega)$, where $\Omega = (0,1) \times \mathbb{R}$ and $\delta > 0$. In this case, we are in the Gaussian regime corresponding to $\alpha = 2$. In [CSZ17], a random measure is constructed from a reference measure $P^{\mathrm{Ref}}_{\Omega_\delta}$ via

$$P^{\omega}_{\Omega_\delta,\lambda,h}(d\sigma) = \frac{\exp\left(\sum_{x\in\Omega_\delta}(\lambda\omega_x + h)\sigma_x\right)}{Z^{\omega}_{\Omega_\delta;\lambda,h}} P^{\mathrm{Ref}}_{\Omega_\delta}(d\sigma). \tag{24}$$

Here $\lambda > 0$, $h \in \mathbb{R}$, and the field σ is given by $\sigma = (\sigma_x)_{x\in\Omega_\delta}$. In the directed polymer setting, we take $\delta = 1/N$ and define the field $\sigma_x = \mathbf{1}_{A_\delta}(x)$, where A_δ is the random subset given by $A_\delta := \left\{\left(\frac{n}{N}, \frac{S_n}{N^{1/2}}\right)\right\}_{n\geq 0}$. Consequently, setting d=1, the deterministic tilt satisfies

$$\sum_{x\in\Omega_\delta} h\sigma_x = \sum_{x\in\Omega_\delta} h\mathbf{1}_{A_\delta}(x) = hN. \tag{25}$$

This should be contrasted with the drift term in (8), which is of the form hS_n. The former depends only on the length of the path, while the latter depends on the endpoint, and hence introduces a genuine drift not previously studied.

The Topology of Convergence The topology of convergence for Theorem 2.2 is that of $C_b([\epsilon, T] \times \mathbb{R})$ with the supremum norm. We implicitly assume that the random environment is extended to $\mathbb{R}$ by taking it to be piecewise constant on the lattice, which, when the lattice is appropriately scaled, converges to a continuous limit.

To show convergence in distribution of the four-parameter process in Theorem 2.2, it suffices to establish convergence of the finite-dimensional distributions (FDDs) and tightness of the field as elements of $C_b([\epsilon, T] \times \mathbb{R})$, see [Bil99] and references therein. We prove tightness in Section 7 and expand on the necessary details in Appendix D and Subsection A.2, which rely on standard SPDE methods.

Core Ideas of the proof The proof of (2.1) relies on the work of [CSZ17], which provides a framework for a family of polynomial chaos expansions to converge to Wiener chaos expansions; see 2.1 and 2.2 in [CSZ17]. The key idea is to linearise the partition function Z_n^{TILT}, this linearisation is given by

$$\mathfrak{Z}_n^{\mathrm{TILT}}(x,\beta) = E_{h_n}^{\mathrm{TILT}}\left[\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})\,\mathbb{1}_{\{S_n=x\}}\right]. \tag{26}$$

Using the binomial theorem, we can expand (26) as a polynomial chaos expansion,

$$E_{h_n}^{\mathrm{TILT}}\left[1+\sum_{k=1}^{n}\beta^k\sum_{\mathbf{i}\in D_k^n}\prod_{i=1}^{k}\omega_{i,S_i}\right] \tag{27}$$

where

$$D_k^n = \left\{\boldsymbol{i} := (i_1,\dots,i_k)\in\mathbb{N}^k : 1\le i_1<\cdots<i_k\le n\right\} \tag{28}$$

Using the Markov property (10) we obtain the full polynomial chaos expansion

$$(27) = 1+\sum_{k=1}^{n}\beta^k\sum_{\mathbf{i}\in D_k^n}\sum_{x\in\mathbb{Z}^k}\prod_{j=1}^{k}\omega(i_j,x_j)P_{h_n}^{\mathrm{TILT}}(i_j-i_{j-1},x_j-x_{j-1}),. \tag{29}$$

Heuristically, if one scales the lattice diffusively and applies the local limit theorem to $P_{h_n}^{\mathrm{TILT}}(i_j-i_{j-1},x_j-x_{j-1})$, see Theorem A.1, then we should obtain the limiting object claimed in (2.1). Formally, this framework is given by [CSZ17] Theorem 2.3, which we will expand upon when in Section 4.

A New Partition Function For Velocity Of a Path We now introduce a new partition function, which describes the polymer observed at a given velocity. The usual partition function records the weight of paths ending at a position x, whereas the part the new partition function records the weights of the path whose velocity is h. We now define a velocity partition function by replacing the simple random walk reference measure with the transformed reference walk. Let

$$t_k = \frac{k}{N}, \qquad H_k^{(N)} = \frac{k}{N}\frac{S_k}{\sqrt{N}}, \tag{30}$$

NOTE: I think this needs changing For $t_0 < t$, set

$$k_0 = \lfloor Nt_0\rfloor, \qquad k = \lfloor Nt\rfloor. \tag{31}$$

We define the velocity point-to-point partition function by

$$U_N(t,h) := E_{t_0,h_0;\,t,h}^{H,N}\left[\exp\left\{\sum_{i=k_0+1}^{k}\Big(\beta\omega\big(i,H_i^{(N)}\big)-\lambda(\beta)\Big)\right\}\right], \tag{32}$$

where $E^{H,N}_{t_0,h_0;\,t,h}$ denotes expectation with respect to the discrete bridge of the transformed walk $H^{(N)}$, started from (t_0, h_0) and conditioned to end at (t, h).

Theorem 2.3. *Under the intermediate disorder scaling*

$$\beta_N = \beta N^{-1/4},$$

the veclocity partition functions U_N converge in distribution to solution of the following SPDE

$$\partial_t U = \frac{1}{2t^2}\partial_{hh}U + \frac{h_0 - h}{t}\partial_h U + \beta t^{-1/2} U\overline{W}. \tag{33}$$

The initial condition is the Dirac delta measure

$$U(t_0, \cdot) = \delta_{h_0}. \tag{34}$$

Replica Overlap In Intermediate Disorder We are also interested in the number of intersections two independant SRW's take in intermediate disorder under polymer measure measure. The main result is the following.

$$I_{n,\beta} := \sum_{x\in\mathbb{Z}} P^{\omega}_{n,\beta_n}(S_n = x)^2. \tag{35}$$

In [CH02], they showed that with unscaled β, the sumability or divergence is equivalent for the polymer being in weak or strong disorder. It is important to note that I_n is a random variable on $(\Omega_\omega, \mathcal{W}, Q)$, and should be viewed as a random sequence $(I_{n,\beta})_{n\in\mathbb{N}}$, with each $I_{n,\beta} \in [0, 1]$. The sequence $(I_{n,\beta})_{n\in\mathbb{N}}$ is a natural object of interest to study, as it is related to the Doob decomposition of the free energy (see [CH02]) and also has an interpretation in terms of replicas; let S_1 and S_2 be two independent SRWs with law $P^{\omega}_{n,\beta}$. On the path space Ω^2_{path} define the product measure $P^{\omega}_{n,\beta} \otimes P^{\omega}_{n,\beta}$. This allows us to write

$$I_{n,\beta} = [P^{\omega}_{n,\beta_n}]^{\otimes 2}(S^1_n = S^2_n) := P^{\omega}_{n,\beta_n} \otimes P^{\omega}_{n,\beta_n}\big(S^1_n = S^2_n\big) \tag{36}$$

and this allows us the interpretation that $\sum_{n=1}^{k} I_n$ is the expected amount of overlap of two polymers in the same fixed environment. We are interested in the intermediate disorder scaling on the endpoint measure and, consequently. The summability of I_n is of major interest, as it characteristics weak and strong disorder.

Theorem 2.4 ([CH02], Theorem 1.1)**.** *Let $d = 1$. Let $\omega betheenvironmentwhichconsistsofi.i.dGaussianrando$*
0, the following are equivalent:

(i) $\displaystyle\sum_{n=1}^{\infty} I_{n,\beta} = \infty \quad a.s.$

(ii) $Z_n(\beta)e^{-\frac{\beta^2}{2}n} \longrightarrow 0 \quad a.s.$

Later [CSY03],uprgraded the result to a general i.i.d. environment.

Their proof relies on the fact that after normalization, the partition function $Z_n(\beta)$ is a martingale, and hence $\log(Z_n(\beta)e^{-\frac{\beta^2}{2}n})$ is a *supermartingale* which admits a Doob decomposition:

$$\log\left(Z_n(\beta)e^{-\frac{\beta^2}{2}n}\right) = M_n + A_n \tag{37}$$

where $M_n$is a martingale with respect to the filtration $\mathcal{D}_n = \sigma\big(\omega(j,x) : j \leq n,\ x \in \mathbb{Z}^d\big)$. and A_n is an increasing adapted process with respect to the same filtration. Then by an application of Ito's lemma one can show that $A_n \asymp I_n$ where $\asymp$ means that there exists constant $C_1(\beta)$ and $C_2(\beta)$ such that $C_1(\beta)A_n \leq I_n \leq C_2 A_n$. One can then apply the strong law for martingales (see [Pet75]) and conclude that as $n \to \infty$ we have that $\log\left(Z_n(\beta)e^{-\frac{\beta^2}{2}n}\right) \to 0$ Q a.s and this provides the result. Since scaling the inverse temperature removes the martingale structure, one cannot hope for any time of almost sure convergence of I_{n,β_n}, however using the theory of [AKQ14b] we can identify the distributional limit in intermediate disorder.

Theorem 2.5 (Endpoint Replica Overlap in Intermediate Disorder)**.** *Let*

$$I_{n,\beta n^{-1/4}} := \sum_{x\in\mathbb{Z}}\Big(P^{\omega}_{n,\beta n^{-1/4}}(S_n = x)\Big)^2$$

Let C or R be a compact set, then for $x \in \mathbb{Z} \cap C$

$$\sqrt{n}\, I_n \xrightarrow{d} \int_C \left(\frac{Z(1,x)}{\displaystyle\int_C Z(1,y)\,dy}\right)^2 dx. \tag{38}$$

where $Z(1,x)$ is the solution to the stochastic heat equation at time $t = 1$. Further we have that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \mathbb{Q}\bigg(I_n \leq \frac{\delta}{\sqrt{n}}\bigg) = 0.$$

Further, if we consider I_{nt}, then we have

$$\sqrt{n}\, I_{nt} \Rightarrow \int_C \left(\frac{Z(t,x)}{\int_C Z(t,y)\,dy}\right)^2 dx. \tag{39}$$

3 Replica Overlap in Intermediate Disorder

Replica Overlap Theory

Proof of Theorem 2.5

Proof. We note that in intermediate disorder we have that

$$I_{n,\beta_n]} = \sum_{x\in\mathbb{Z}/\sqrt{n}} P^{\omega}_{\beta_n,n}\big(S_n = x\sqrt{n}\big)^2$$
$$= \sum_{x\in\mathbb{Z}/\sqrt{n}} \frac{Z_n^2(x\sqrt{n},\beta_n)}{Z_n^2(\beta_n)} \tag{40}$$

Lemma 3.1. *Let $0<\epsilon \ll 1/2$. Let C be a compact set. Consider the endpoint measure as* ***NOTE:*** *you need to write down the endpoint measure somewhere. Then*

$$\sum_{x\in\mathbb{Z}/\sqrt{n}\cap C} \frac{nZ_n^2(x\sqrt{n},\beta_n)}{\sqrt{n}Z_n^2(\beta_n)} = \int_C \frac{nZ_n^2(x\sqrt{n},\beta_n)}{Z_n^2(\beta_n)}dx + O(n^{-1/2-\epsilon}), \tag{41}$$

as $n\to\infty$.

The proof relies of a straightforward Riemann approximation to the sum, with the error coming from the modulus of continuity of the partition function. The modulus of continuity estimates were done by [AKQ14b] in Appendix B. However we will give a brief overview of their method and explain the adaptation to our case.

Define $z_n(x,t,\beta_n) := \sqrt{n}\,\mathfrak{z}\big(nt, x\sqrt{n}, \beta n^{-1/4}\big)$, which is the modified point to point partition function with intermediate disorder scaling (see Appendix B of [AKQ14b] or Proposition A2). If t is fixed we will remove it from the notation and replace it with notation $z_n(x,\beta_n)$. By defining the random environment **NOTE:** should I write this change in enviroment in the same manner as [AKQ14b] (see Proposition 5.4 and therein), we are able to interchange $nZ_n^2(x\sqrt{n},\beta_n)$ with $z_n^2(x,\beta_n)$ in the statement of Lemma 1.

By considering the sum over x such that $x\in\mathbb{Z}/\sqrt{n}$, we see that the LHS of Lemma 1 is an approximation to the Riemann integral. We have a slight annoyance as we need to deal with two replicas of the SRW and hence we need to deal with the endpoint measure squared. Our approach is to apply the Garsia–Rumsey–Rodemich (GRR) inequality to the spatial sample paths of $z_n^2(x\sqrt{n},\beta_n)$, which will provide modulus of continuity estimates for $x\in\mathbb{R}$ and time $[\epsilon,T]$, than can be used to approximate the Riemann integral in Lemma 1; note that $Z_n(\beta_n)$ is just a finite random constant which does effect the Holder exponent. For a function $f:\mathbb{R}\to\mathbb{R}$, define the α-Hölder modulus of continuity $\omega_f(\delta)$, to be the constant given by

$$\sup_{|x-y|\le\delta} |f(x)-f(y)| := \omega_f(\delta). \tag{42}$$

It was proven in Theorem B.2 that there exists a version of $z_n(x,t,\beta_n)$ which has continuous sample paths in both space and time. Therefore, $z_n^2(x,t,\beta_n)$ also has continuous sample paths in space and time and we apply the continuous spatial and time sample paths to the statement of the Garsia–Rumsey–Rodemich (GRR). Before doing so, we consider the elementary inequalities. For $M > 12$ **NOTE:** I have written 12, but need to check if it could be 6,

$$\begin{aligned}
&\mathbb{Q}\big[|z_n^2(x,t) - z_n^2(y,s)|^M\big] \\
&\leq \mathbb{Q}\big[|z_n(x,t) - z_n(y,s)|^{2M}\big]^{1/2}\mathbb{Q}\big[|z_n(x,t) + z_n(y,s)|^{2M}\big]^{1/2} \\
&\leq \mathbb{Q}\big[|z_n(x,t) - z_n(y,s)|^{2M}\big]^{1/2}\Big(\mathbb{Q}\big[|z_n(x,t)|^{2M}\big]^{1/2} + \mathbb{Q}\big[|z_n(y,s)|^{2M}\big]^{1/2}\Big),
\end{aligned} \tag{43}$$

Lemma B1 of [AKQ14b] tells us that for $M > 2$, there exists a constant C_M such that for any $t > 0$ and $x \in \mathbb{Z}$ and $n \geq 1$,

$$\mathbb{E}[z_n(x,t)^M] \leq C_M \rho(x,t)^M. \tag{44}$$

where $(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2}$ is the standard heat kernel

Therefore applying inequality (44) to (43), we get that for $M > 12$

$$\begin{aligned}
&\mathbb{Q}\big[|z_n^2(x,t) - z_n^2(y,s)|^M\big] \\
&\leq C_M \mathbb{Q}\big[|z_n(x,t) - z_n(y,s)|^{2M}\big]^{1/2} \rho(x,t)^{M/2}\rho(y,s)^{M/2}
\end{aligned} \tag{45}$$

The modulus of continuity estimates for $z_n(x,t)$ in [AKQ14b] (Lemma B.2) further show that for $M > 6$,

$$\begin{aligned}
\mathbb{Q}\big[|z_n^2(x,t) - z_n^2(y,s)|^M\big] &\leq C_M(|x-y| + |s-t|)^{M/4-2}\rho(x,t)^{M/2}\rho(y,s)^{M/2} \\
&= C_M(|x-y| + |s-t|)^{1+\alpha}\rho(x,t)^{M/2}\rho(y,s)^{M/2},
\end{aligned} \tag{46}$$

for $n \geq 1$ and any $M > 12$ to ensure that $\alpha := M/4 - 3$ is strictly positive. By an application of Fubini's theorem, taking $\phi(x) = x^M$, $\varphi(x) = x^{\gamma/M}$ for any even $M > 12$. Checking the hypothesis of GRR to the sample paths of $z^2(x,t)$

gives us

$$\begin{aligned}
&\mathbb{Q}\left[\int\int_{\substack{x,y\in\mathbb{R}/\sqrt{n}\cap C\\ t,s\in[\epsilon,T]}}\phi\left(\frac{|z_n^2(x,t)-z_n^2(y,s)|}{\varphi(|x-y|+|s-t|)}\right)\,dx\,dy\,dt\,ds\right]\\
&\le\int\int_{\substack{x,y\in\mathbb{R}/\sqrt{n}\cap C\\ t,s\in[\epsilon,T]}}\frac{C_M\left(|x-y|+|s-t|\right)^{\frac{M}{4}-2}\rho(x,t)^{M/2}\rho(y,s)^{M/2}}{(|x-y|+|s-t|)^{\gamma}}\,dx\,dy\,dt\,ds\\
&=\int\int_{\substack{x,y\in\mathbb{R}/\sqrt{n}\cap C\\ t,s\in[\epsilon,T]}}C_M\left(|x-y|+|s-t|\right)^{\frac{M}{4}-2-\gamma}\rho(x,t)^{M/2}\,\rho(y,s)^{M/2}\,dx\,dy\,dt\,ds\\
&\le\int\int_{\substack{x,y\in\mathbb{Z}/\sqrt{n}\\ t,s\in[\epsilon,T]}}C_M\left(|x-y|+|s-t|\right)^{\frac{M}{4}-2-\gamma}\rho(x,t)^{M/2}\,\rho(y,s)^{M/2}\,dx\,dy\,dt\,ds\\
&\le C_{M,\gamma}.
\end{aligned}\tag{47}$$

provided that $M/4-2-\gamma>-1$, which holds true for any even $M>12$ and $\gamma<M/4-1$. Therefore we have that

$$\mathbb{Q}\left[\int\int_{\substack{x,y\in C\\ t,s\in[\kappa,T]}}\phi\left(\frac{z_n^2(x,t)-z_n^2(y,s)}{\varphi(|x-y|+|s-t|)}\right)\,dx\,dy\,dt\,ds\right]=B_{M,t}(\omega)\tag{48}$$

where $B_{M,t}(\omega)$ is a random constant. By an application of Chebyshev's inequality

$$P(B_{M,t}(\omega)>B)\le C_{M,t}/B\tag{49}$$

So for all $t\in[\kappa,T]$ and $x,y\in\mathbb{Z}\sqrt{n}\cap C$, by the GRR inequality

$$\sup_{\substack{x,y\in(\mathbb{Z}/\sqrt{n})\cap C\\ |x-y|\le h\\ t,s\in[\kappa,T]\\ |s-t|\le\delta}}|z_n^2(x,t)-z_n^2(y,s)|\le C_{M,\gamma}B^{1/M}(|h|+|\delta|)^{(\gamma-4)/M}\tag{50}$$

where $(\gamma-4)/M>0$ when $\gamma>4$. Recalling that $\gamma<M/4-1$ we have that $(\gamma-4)/M\le\frac{M/4-5}{M}=1/4-5/M$, so taking $M>6$ large and even provides a modulus of continuity for any $\alpha<1/4$. $\square$

We need to provide a Riemann approximation of the sum (40). Let C be a compact set and consider a partition of this space given by

$$\bigcup_{1\le k\le\sqrt{n}}\left(\frac{k-1}{\sqrt{n}},\frac{k}{\sqrt{n}}\right):=\bigcup_{1\le k\le\sqrt{n}}J_k,\tag{51}$$

and note that for each k we have $|J_k| = \frac{1}{\sqrt{n}}$. We consider the following difference

$$\left| \frac{1}{\sqrt{n}}\, z_n^2\left(\frac{k}{\sqrt{n}}\right) - \int_{J_k} z_n^2(x)\, dx \right| \leq |J_k|\, \omega_{z_n^2}(\delta) \tag{52}$$

where $\delta := x - k/\sqrt{n}$, so that $|\delta| \leq 1/\sqrt{n}$ for all $x \in J_k$. Using the modulus of continuity estimate given in (50) we obtain that

$$\omega_{z_n^2}(\delta) \;\leq\; C\,|\delta|^\alpha \;\leq\; C\left(\frac{1}{\sqrt{n}}\right)^\alpha \tag{53}$$

for any $\alpha < 1/4$. Summing both sides of (52) over $1 \leq k \leq \sqrt{n}$ gives

$$\left| \frac{1}{\sqrt{n}} \sum_{k=1}^{\sqrt{n}} z_n^2\left(\frac{k}{\sqrt{n}}\right) - \int_C z_n^2(x)\, dx \right| \leq \sqrt{n} \cdot \frac{1}{\sqrt{n}} \cdot C\left(\frac{1}{\sqrt{n}}\right)^\alpha = C \cdot n^{-\alpha/2} \tag{54}$$

which tends to zero as $n \to \infty$ for any $\alpha > 0$.

Define a functional $\varphi : C_b[C] \to \mathbb{R}$ by $\varphi(f) = \int_C f^2(x)\, dx$. It is easy to check that this φ is well-defined, and note that φ is continuous map since

$$|F(f) - F(g)| \leq |C| |(f-g)| \|f\|_\infty \|g\|_\infty. \tag{55}$$

Therefore by the local limit theorem of [AKQ14b] (see (16) - which includes tightness and convergence of the finite dimensional distributions) we have that

$$\varphi\left(\frac{\sqrt{n}\, Z_n(x\sqrt{n}, \beta_n)}{2\, Z(\beta_n)}\right) \to \varphi\left(\frac{e^{A_{2\sqrt{\beta}}(x)}\, \rho(1,x)}{\mathcal{Z}_{\sqrt{2}\beta}}\right) \tag{56}$$

as $n \to \infty$ where $e^{A_{2\sqrt{\beta}}(x)}$ is the crossover distribution (see Proposition A1). Theorem 1 part (5) of [Flo14] tells us that the numerator of (56) is strictly positive as it is the solution of Stochastic Heat Equation (SHE),

$$\begin{aligned} \partial_t \mathcal{Z} &= \tfrac{1}{2}\Delta \mathcal{Z} + \beta \mathcal{Z} \xi, \\ \mathcal{Z}(0,x) &= \delta_0(x). \end{aligned} \tag{57}$$

Define

$$R := \frac{R_1}{R_2} := \frac{\displaystyle\int_C u(1,x)^2\, dx}{\left(\displaystyle\int_C u(1,y)\, dy\right)^2}. \tag{58}$$

We claim that R is finite and positive almost surely. We note that by Theorem C.1, we know that both R_1 and R_2 are positive almost surely, so what remains to show is that both R_1 and R_2 are finite almost surely. We consider

$$\mathbb{E}[R_1] = \mathbb{E}\left[\int_C u(1,x)^2\,dx\right] \tag{59}$$
$$= \int_{\mathbb{R}} \mathbb{E}\left[u(1,x)^2\right]\,dx$$
$$\leq \int_{\mathbb{R}} \frac{Ce^{-x^2}e^{1/4}}{2\pi}\,dx < \infty.$$

where we have used an application of Fubini and the results of [CD15, Eq. (2.30)]. see Appendix C). A similar calculation shows that

$$\mathbb{E}[R_2] \leq \mathbb{E}\left[\left(\int_{\mathbb{R}} u(1,x)\,dx\right)^2\right] \tag{60}$$
$$= \mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} u(1,x)u(1,y)\,dx\,dy\right] \tag{61}$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\left[u(1,x)u(1,y)\right]\,dx\,dy \tag{62}$$
$$\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{Ce^{-x^2}e^{-y^2}e^{1/2}}{4\pi^2}\,dx\,dy < \infty. \tag{63}$$

where we have again used Fubini's theorem, Hölder's inequality, and the moment estimate in [CD15, Eq. (2.30)]. **NOTE:** Double check this citation and the constants. Note we can also obtain bounds on R, these are given by

$$R = \frac{R_1}{R_2} \geq \int_{-L}^{L} \frac{u(1,x)^2}{R_2}\,dx \tag{64}$$
$$\geq \frac{1}{2LR_2}\left(\int_{-L}^{L} u(1,x)\,dx\right)^2 := \frac{T_L^2}{2LR_2} \tag{65}$$

Now conisder a truncation parameter M and note that

$$\mathbb{P}(R \leq \delta) \leq \mathbb{P}\left(\frac{T_L^2}{2LR_2} \leq \delta\right) \tag{66}$$
$$\leq \mathbb{P}\left(T_L \leq \sqrt{2L\delta M}\right) + \mathbb{P}(R_2 > M). \tag{67}$$

The last inequality can be justified by noting restricting to the event $\{R_2 \leq M\}$, the inequality $T_L \leq \sqrt{2L\delta R_2}$ implies $T_L \leq \sqrt{2L\delta M}$. Therefore, $\left\{\frac{T_L^2}{2LR_2} \leq \delta\right\} \subseteq \left\{T_L \leq \sqrt{2L\delta M}\right\} \cup \{R_2 > M\}$. An application of Markov's inequality shows that $\mathbb{P}(R_2 > M) \to 0$ as $M \to \infty$. By an application of the Portmanteau theorem together with Theorem C.1, we obtain

$$\lim_{\delta\downarrow 0}\mathbb{P}\Big(T_L \le \sqrt{2L\delta M}\Big) = 0. \tag{68}$$

Therefore Since $\sqrt{n}I_n \xrightarrow{d} R$, then by another application Portmanteau we have that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} Q\left(I_n \le \frac{\delta}{\sqrt{n}}\right) = 0. \tag{69}$$

Proof of ??

Proof. The first part of the proof follows directly from the approximation of the Riemann integral given in Theorem 3.1, together with the same error bound obtained via a GRR-style argument, as outlined in the proof of Theorem 2.5. Using Theorem B.1, we obtain

$$\sqrt{n}\, I_{nt} \Rightarrow \int_{\mathbb{R}} \left(\frac{Z(t,x)}{\int_{\mathbb{R}} Z(t,y)\,dy}\right)^2 dx.$$

For the second part, define $X_n(s) := \sqrt{n}\, I_{ns}$ for $s \in [\epsilon, t]$. By the above and the GRR estimates, $(X_n)_{n\ge 1}$ is tight in $C([\epsilon,t])$, and hence $X_n \Rightarrow X \quad \text{in } C([\epsilon,t])$, where

$$X(s) = \int_{\mathbb{R}} \left(\frac{Z(s,x)}{\int_{\mathbb{R}} Z(s,y)\,dy}\right)^2 dx. \tag{70}$$

Moreover,

$$\frac{1}{\sqrt{n}} \sum_{k=\lfloor n\epsilon\rfloor}^{\lfloor nt\rfloor} I_k = \int_\epsilon^t X_n(s)\,ds + o(1),$$

where the error term vanishes as $n\to\infty$, by the modulus of continuity estimates established in **??**.

Since the map $f \mapsto \int_\epsilon^t f(s)\,ds$ is continuous on $C([\epsilon,t])$, the continuous mapping theorem yields

$$\int_\epsilon^t X_n(s)\,ds \Rightarrow \int_\epsilon^t X(s)\,ds,$$

which gives the desired result. ☐

4 Convergence Of The Finite Dimensional Distributions For Partition Function With Drift

Multiple Stochastic Integrals We will heavily rely on the theory of multiple stochastic integrals, which originates in the work of [?]. A white noise W is a collection of jointly Gaussian, mean-zero random variables defined on a common probability space $(\Omega_W, \mathcal{F}_W, \mathbb{P})$, indexed by Borel sets of finite measure.

$$W = \{W_A : A \in \mathcal{B},\ \mu(A) < \infty\} \tag{71}$$

where $\mathcal{B}$ denotes the Borel σ-algebra and μ is Lebesgue measure. The collection satisfies

$$\mathbb{E}[W_A] = 0, \qquad \mathbb{E}[W_A W_B] = \mu(A \cap B). \tag{72}$$

Note that if A and B are disjoint, then W_A and W_B are independent random variables. We write $W^{\otimes k}$ for the k-fold product noise, and for $g(\mathbf{t}, \mathbf{x}) \in L^2\big((0,1)^k \times \mathbb{R}^k\big)$, we define

$$W^{\otimes k}(g) := \int g(\mathbf{t}, \mathbf{x})\, W^{\otimes k}(d\mathbf{t}\, d\mathbf{x}) = \int g(t_1, x_1, \ldots, t_k, x_k)\, W(dt_1, dx_1) \cdots W(dt_k, dx_k). \tag{73}$$

The full theory of multiple stochastic integrals can be found in [?] and [Jan97], including formal constructions using simple functions and basic properties such as linearity. However, we note that since white noise is independent on disjoint sets, the underlying stochastic integral is invariant under permutations of the variables. Consequently, multiple stochastic integrals depend only on the symmetric part of the integrand, and it is therefore natural to define them on the space of symmetric functions $L^2_{\text{sym}}\big((0,1)^k \times \mathbb{R}^k\big)$. On the diagonal, additional contributions corresponding to Itô corrections arise, which are avoided by restricting the integration to the simplex $0 < t_1 < \cdots < t_k < 1$. This leads to the natural construction of multiple stochastic integrals.

$$\int_{[0,1]^k} \int_{\mathbb{R}^k} \operatorname{Sym} g(\mathbf{t}, \mathbf{x})\, W^{\otimes k}(d\mathbf{t}\, d\mathbf{x}) = k! \int_{\Delta_k} \int_{\mathbb{R}^k} g(\mathbf{t}, \mathbf{x})\, W^{\otimes k}(d\mathbf{t}\, d\mathbf{x}), \tag{74}$$

where $\Delta_k = \{0 < t_1 < \cdots < t_k < 1\}$. For each square integrable random variable X in $(\Omega_W, \mathcal{F}_W, \mathbb{P})$, there is a unique decomposition given by

$$X = \sum_{k=0}^{\infty} W^{\otimes k}(g_k) \tag{75}$$

where g_k is a sequence of functions in $L^2_{\text{sym}}\big((0,1)^k \times \mathbb{R}^k\big)$, one should take $g_0 = 0$. For each $k \geq 1$, we have $\mathbb{E}\big[W^{\otimes k}(g_k)\big] = 0$, and consequently, from (75), it follows that $\mathbb{E}[X] = 0$. There is also an underlying orthogonality relationship given, that is for $k \neq l$ we have

$$\mathbb{E}\big[W^{\otimes k}(g_k)\, W^{\otimes l}(g_l)\big] = 0. \tag{76}$$

This can be seen by noting that taking expectations as in (76) produces $k+l$ copies of white noise. Since white noise is centered and Gaussian, non-zero contributions arise only when all noise terms can be paired. If $k \neq l$, then the total number of factors is $k+l$, and it is not possible to pair all terms. Hence, using independence and the Gaussian structure, we conclude (76). When $k = l$, all noise terms can be paired, and using the covariance structure of white noise,

$$\mathbb{E}[W(dt,dx)\, W(dt,dx)] = dt\, dx \tag{77}$$

we obtain

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k!\, \|g_k\|^2_{L^2_{\text{sym}}\left((0,1)^k \times \mathbb{R}^k\right)}. \tag{78}$$

We also recall the structure of the (Gaussian) Fock space. Let $\mathcal{H} := L^2\big([0,1] \times \mathbb{R}\big)$. The symmetric Fock space over $\mathcal{H}$ is defined as

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{k=0}^{\infty} \mathcal{H}^{\otimes_{\text{sym}} k}, \tag{79}$$

where $\mathcal{H}^{\otimes_{\text{sym}} k}$ denotes the k-fold symmetric tensor product of $\mathcal{H}$, with the convention $\mathcal{H}^{\otimes 0} = \mathbb{R}$. An element $g \in \mathcal{F}(\mathcal{H})$ can be written as a sequence $g = (g_0, g_1, \dots)$ with $g_k \in L^2_{\text{sym}}\big(([0,1] \times \mathbb{R})^k\big)$, and norm $\|g\|^2_{\mathcal{F}} = \sum_{k=0}^{\infty} k!\, \|g_k\|^2_{L^2}$. The map $I : \mathcal{F}(\mathcal{H}) \to L^2(\Omega)$ defined by

$$I(g) = \sum_{k=0}^{\infty} W^{\otimes k}(g_k) \tag{80}$$

is an isometry, where $I_k(g_k)$ denotes the k-th multiple stochastic integral. We call (75) the *Wiener Chaos Expansion* provided that the sum in (78) is finite.

Recall Theorem A.1, we have the local limit theorem

$$P^{\text{TILT}}_{h_n}(S_n = x) = \frac{2}{\sqrt{2\pi n}} \exp\left(- \frac{2\left(x - \frac{h\sqrt{n}}{2}\right)^2}{n} \right) + O\Big(n^{-1+o(1)}\Big). \tag{81}$$

Under diffusive scaling $x \mapsto x\sqrt{n}$ and $t \mapsto nt$, the exponential term becomes

$$\frac{2}{\sqrt{2\pi nt}} \exp\left(-\frac{2\left(x\sqrt{n}-\frac{h\sqrt{n}}{2}\right)^2}{nt}\right) = \frac{2}{\sqrt{2\pi nt}} \exp\left(-\frac{2\left(x-\frac{h}{2}\right)^2}{t}\right). \tag{82}$$

Hence, after multiplying by $\sqrt{n}$, it is natural to define

$$\rho^{\mathrm{TILT}}(t,x) := \frac{2}{\sqrt{2\pi t}} \exp\left(-\frac{2\left(x-\frac{h}{2}\right)^2}{t}\right). \tag{83}$$

and their corresponding k fold transition probabilities.

Lemma 4.1. *Let $h \in \mathbb{R}$. For $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{t} \in \Delta_k$ (recall the defintion from Theorem 2.1), the k-fold transition probabilities, $\rho_k(\mathbf{t}, \mathbf{x})$, are given by*

$$\rho_k^{\mathrm{TILT}}(\mathbf{t},\mathbf{x}) = \prod_{j=1}^{k} \rho^{\mathrm{TILT}}(t_j - t_{j-1},\, x_j - x_{j-1} - h(t_j - t_{j-1}))\,. \tag{84}$$

Proof. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and define the Brownian motion with drift h by $\tilde{B}_t = B_t + ht$. Fix $0 = t_0 < t_1 < \cdots < t_k$. Then the increments $\tilde{B}_{t_1} - \tilde{B}_{t_0},\ \tilde{B}_{t_2} - \tilde{B}_{t_1},\ \ldots,\ \tilde{B}_{t_k} - \tilde{B}_{t_{k-1}}$ are independent. Moreover, for each $1 \leq j \leq k$, $\tilde{B}_{t_j} - \tilde{B}_{t_{j-1}} = (B_{t_j} - B_{t_{j-1}}) + h(t_j - t_{j-1})$, so $\tilde{B}_{t_j} - \tilde{B}_{t_{j-1}} \sim N\big(h(t_j - t_{j-1}),\, t_j - t_{j-1}\big)$. Hence the vector of increments $(y_1, \ldots, y_k)$ has density

$$\prod_{j=1}^{k} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \exp\left(-\frac{(y_j - h(t_j - t_{j-1}))^2}{2(t_j - t_{j-1})}\right). \tag{85}$$

Now set $x_j = y_1 + \cdots + y_j$ for $1 \leq j \leq k$, with $x_0 = 0$, so that $y_j = x_j - x_{j-1}$. This change of variables has Jacobian equal to 1, and therefore

$$\rho_k^{\mathrm{TILT}}(\mathbf{t},\mathbf{x}) = \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \exp\left(-\frac{(x_j - x_{j-1} - h(t_j - t_{j-1}))^2}{2(t_j - t_{j-1})}\right). \tag{86}$$

$\square$

We can now show that the random variable

$$X = \sum_{k=0}^{\infty} W^{\otimes k}(\rho_k^{\mathrm{TILT}}) \tag{87}$$

is a well-defined Wiener chaos expansion by showing that (78) is finite. Note that we may use the identity

$$\left(\rho_k^{\mathrm{TILT}}(\mathbf{t},\mathbf{x})\right)^2 = \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}}\, p\Big(t_j - t_{j-1}, \sqrt{2}\left(x_j - x_{j-1} - h(t_j - t_{j-1})\right)\Big). \tag{88}$$

Integrating first over $\mathbb{R}^k$ and then over Δ_k, we obtain

$$\int_{\Delta_k}\int_{\mathbb{R}^k}\left(\rho_k^{\mathrm{TILT}}(\mathbf{t},\mathbf{x})\right)^2 d\mathbf{x}\,d\mathbf{t} = \int_{\Delta_k}\prod_{j=1}^{k}\frac{1}{2\sqrt{\pi(t_j-t_{j-1})}}\,d\mathbf{t}, \tag{89}$$

where we have used a change of variables to remove the drift. The remaining term can be identified with a Dirichlet integral, yielding a ratio of Gamma functions. Using Subsection E.4 the resulting expression behaves like $\Gamma(k/2+1)^{-1}$, which decays sufficiently fast by Sterlings' approximation to ensure that the sum of the L^2 norms is summable.

U-Statisitics U-statistics are sums of the form

$$\sum_{\mathbf{i}\in D_n^k}\sum_{\mathbf{x}\in\mathbb{Z}^k}\mathbb{1}_{\{\mathbf{i}\leftrightarrow\mathbf{x}\}}\,g(\mathbf{i},\mathbf{x})\,\omega(\mathbf{i},\mathbf{x}), \tag{90}$$

where we refer to $g(\mathbf{i},\mathbf{x})$ as the weight function. We discretise $g(\mathbf{i},\mathbf{x})$ by averaging it over diffusively scaled rectangles. Let $\mathcal{F}_n$ denote the collection of rectangles of the form

$$\mathcal{F}_n := \left\{\left(\frac{\mathbf{i}-\mathbf{1}}{n},\frac{\mathbf{i}}{n}\right]\times\left(\frac{\mathbf{x}-\mathbf{1}}{\sqrt{n}},\frac{\mathbf{x}+\mathbf{1}}{\sqrt{n}}\right]\;\middle|\;\mathbf{i}\in D_n^k,\ \mathbf{x}\in\mathbb{Z}^k,\ \mathbf{i}\leftrightarrow\mathbf{x}\right\}. \tag{91}$$

Then, for $g\in L^2([0,1]^k\times\mathbb{R}^k)$, define

$$\overline{g}_n(\mathbf{t},\mathbf{x}) = \frac{1}{|R|}\int_R g(\mathbf{s},\mathbf{y})\,d\mathbf{s}\,d\mathbf{y}, \qquad (\mathbf{t},\mathbf{x})\in R\in\mathcal{F}_n. \tag{92}$$

This may be viewed as the conditional expectation of g with respect to the σ-algebra generated by the partition $\mathcal{F}_n$. In particular, for each $R\in\mathcal{F}_n$, the function $\overline{g}_n(\mathbf{t},\mathbf{x})$ is constant on R. Then consider the following U-statistic

$$S_k^n(g) := \sum_{\mathbf{i}\in E_n^k}\sum_{\mathbf{x}\in\mathbb{Z}^k}\mathbb{1}_{\{\mathbf{i}\leftrightarrow\mathbf{x}\}}\,\overline{g}_n(\mathbf{i},\mathbf{x})\,\omega(\mathbf{i},\mathbf{x}) \tag{93}$$

We need the following standard fact from Hilbert space theory: if $(X_k)_{k\geq 0}$ is a sequence of random variables in L^2 such that $\sum_{k=0}^{\infty}\mathbb{E}[X_k^2]<\infty$, then $\sum_{k=0}^{\infty}X_k$ converges in L^2.

We state Theorem 4.3 and Lemma 4.4 of [AKQ14b], which in turn is heavily influenced by Theorem 11.16 of [Jan97]. We state Theorem 4.3 and Lemma 4.4 of [AKQ14b], which in turn is heavily influenced by Theorem 11.16 of [Jan97].

Theorem 4.2 (Theorem 4.1 in [AKQ14b]). *The map S_k^n is linear in the sense that for all $\alpha_1,\dots,\alpha_m$ and $g_1,\dots,g_m\in L^2([0,1]^k\times\mathbb{R}^k)$, we have*

$$\sum_{j=1}^{m}\alpha_j S_k^n(g_j) = S_k^n\left(\sum_{j=1}^{m}\alpha_j g_j\right) \tag{94}$$

with probability one.

Further, for each g_k, the random variable $S_k^n(g_k)$ has mean zero and the family is orthogonal, in the sense that for $k \neq l$,

$$\mathbb{E}_Q[S_k^n(g_k)\, S_l^n(g_l)] = 0. \tag{95}$$

If $k = l$, then

$$\mathbb{E}_Q\left[S_k^n(g_k)^2\right] \leq n^{3k/2} \|\overline{g}_n\|^2_{L^2([0,1]^k \times \mathbb{R}^k)} \leq n^{3k/2} \|g\|^2_{L^2([0,1]^k \times \mathbb{R}^k)}. \tag{96}$$

Theorem 4.3 (Theorem 4.3 and Lemma 4.4 in [AKQ14b])**.** *Let $g \in L^2([0,1]^k \times \mathbb{R}^k)$. Then, as $n \to \infty$,*

$$n^{-3k/4} S_k^n(g) \xrightarrow{(d)} \int_{[0,1]^k} \int_{\mathbb{R}^k} g(\mathbf{t}, \mathbf{x})\, W^{\otimes k}(d\mathbf{t}, d\mathbf{x}). \tag{97}$$

Moreover, for any $\mathbf{g} = (g_0, g_1, \dots) \in \bigoplus_{k=0}^{\infty} L^2_{\mathrm{sym}}([0,1]^k \times \mathbb{R}^k)$, as $n \to \infty$,

$$\sum_{k=0}^{n} n^{-3k/4} S_k^n(g_k) \xrightarrow{(d)} \sum_{k=0}^{\infty} \int_{[0,1]^k} \int_{\mathbb{R}^k} g_k(\mathbf{t}, \mathbf{x})\, W^{\otimes k}(d\mathbf{t}, d\mathbf{x}) =: I(\mathbf{g}). \tag{98}$$

The main proof strategy is the same as [AKQ14b], that is we will need to transfer our results from $\mathfrak{Z}_n^{\mathrm{TILT}}(\beta)$ to $Z_n^{\mathrm{TILT}}(\beta)$ by considering a perturbation of the random environment **NOTE:** which section. **NOTE:** should this bit be here? We are interested in a random environment $\tilde{\omega}_n(i,x)$ which only has mean asymptotically to zero and variance asymptotically to 1, that is the perturbative random environment $\omega_n(\tilde{i}, x)$ has an explicit dependence on n. Again we can define a weighted U-Stasitic for $g \in L^2([0,1]^k \times \mathbb{R}^k)$.

$$\mathbf{S}_k^n(g; \tilde{\omega}_n) = 2^{k/2} \sum_{\mathbf{i} \in E_n^k} \sum_{\mathbf{x} \in \mathbb{Z}^k} \mathbb{1}_{\{\mathbf{i} \leftrightarrow \mathbf{x}\}}\, \overline{g}\left(\frac{\mathbf{i}}{n}, \frac{\mathbf{x}}{\sqrt{n}}\right) \tilde{\omega}_n(\mathbf{i}, \mathbf{x}) \tag{99}$$

We will need analogous of Theorem 4.1 and Theorem 4.2 where our random environment is now a function of n.

Theorem 4.4. *Assume that the environment $\tilde{\omega}_n(\mathbf{i}, \mathbf{x})$ satisfies*

$$Q\big(\tilde{\omega}_n(\mathbf{i}, \mathbf{x})\big) = 0, \qquad Q\big(\tilde{\omega}_n^2(\mathbf{i}, \mathbf{x})\big) = 1 + o(1),$$

uniformly in $(\mathbf{i}, \mathbf{x})$. Then the result of (4.3) holds with $S_k^n(g)$ replaced by $S_k^n(g; \tilde{\omega}_n)$.

Moreover, suppose that $\mathbf{g} = (g_0, g_1, \dots) \in \bigoplus_{k=0}^{\infty} L^2_{\mathrm{sym}}([0,1]^k \times \mathbb{R}^k)$. If

$$\lim_{N \to \infty} \limsup_{n \to \infty} \sum_{k=N}^{\infty} Q(\tilde{\omega}_n^2) \|g_k\|^2_{L^2([0,1]^k \times \mathbb{R}^k)} = 0, \tag{100}$$

Then line (98) also holds.

Convergence of $\mathfrak{Z}_n^{\mathrm{TILT}}(x,\beta)$. We recall from (2.1) that

$$D_k^n = \left\{ \boldsymbol{i} := (i_1, \ldots, i_k) \in \mathbb{N}^k : 1 \leq i_1 < \cdots < i_k \leq n \right\}. \tag{101}$$

We will also need an unordered version of D_k^n, which is given by

$$E_k^n = \left\{ \boldsymbol{i} \in [n]^k : i_j \neq i_\ell \text{ for } j \neq \ell \right\}. \tag{102}$$

We prove convergence of $\mathfrak{Z}_n^{\mathrm{TILT}}(x,\beta)$ using the theory of U-statistics. We then transfer these results to the standard partition function in the next section. Throughout, we assume that the random environment has mean of zero and unit variance. Recall the definition of $P_k^{\mathrm{TILT}}(\mathbf{t},\mathbf{x})$ from (**??**) **NOTE:** I need a reference here. We extend this function to $[0,1]^k \times \mathbb{R}^k$ as follows. Consider the random variable

$$X_n(t) := \frac{S_{\lfloor nt \rfloor} + U_{\lfloor nt \rfloor}}{\sqrt{n}}, \tag{103}$$

where $(U_i)_{i\geq 1}$ are Unif$(-1,1)$ i.i.d random variables and S_{nt} is the diffusivly scaled SRW with law P_h^{TILT}. On the event $\{S_i = x\}$, we obtain $\frac{x+U_i}{\sqrt{n}}$ which is uniformly distributed on the interval $\left(\frac{x-1}{\sqrt{n}}, \frac{x+1}{\sqrt{n}}\right)$, with density $\sqrt{n}/2$. This naturally extends to k dimensions, where, by independence, the density will $(\sqrt{n}/2)^k$.

$$p_k^n\left(\frac{\mathbf{i}}{n}, \frac{\mathbf{x}}{\sqrt{n}}\right) := 2^{-k} n^{k/2} p_k(\mathbf{i},\mathbf{x})\, \mathbb{1}_{\{\mathbf{i}\in D_k^n\}}. \tag{104}$$

Define for $1 \leq k \leq n$,

$$p_{k,n}^{\mathrm{TILT}}(\mathbf{t},\mathbf{x}) := \overline{p}_k^{\mathrm{TILT}}\left(\lceil n\mathbf{t} \rceil, \sqrt{n}\,\mathbf{x}\right)\, \mathbb{1}_{\{\lceil n\mathbf{t}\rceil \in D_n^k\}}. \tag{105}$$

Note that when $k > n$, the condition that $\lceil n\mathbf{t} \rceil \in D_n^k$ becomes inpossible and consequently $p_{k,n}^{\mathrm{TILT}}(\mathbf{t},\mathbf{x}) = 0$ for $k > n$. **NOTE:** You need to say something about the fact that this is a smoothing of the Kernel for this to make sense. Under diffusive scaling, we claim that

$$\overline{p}_{k,n}^{\mathrm{TILT}}\left(\frac{\mathbf{i}}{n}, \frac{\mathbf{x}}{\sqrt{n}}\right) = 2^{-k}\overline{p}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x})\mathbb{1}_{\{\mathbf{i}\in D_n^k\}}. \tag{106}$$

Therefore, by using the definition of (93), we have that

$$S_k^n\left(\overline{p}_{k,n}^{\mathrm{TILT}}\right) = 2^{-k/2} \sum_{\mathbf{i}\in D_n^k} \sum_{\mathbf{x}\in\mathbb{Z}^k} \overline{p}_k^{\mathrm{TILT}}(\mathbf{i},\mathbf{x})\,\omega(\mathbf{i},\mathbf{x}) \tag{107}$$

Recalling the definition of the modified point-to-line partition function given in (29)

$$
\begin{aligned}
\mathfrak{Z}_n^{\mathrm{TILT}}(\beta) &= \sum_{k=0}^{n} \beta_n^k \sum_{\mathbf{i}\in D_k^n} \sum_{\mathbf{x}\in\mathbb{Z}^k} \left(\prod_{j=1}^{k} \omega(i_j, x_j) p_k^{\mathrm{TILT}}(i_j - i_{j-1},\, x_j - x_{j-1}) \right) \\
&= \sum_{k=0}^{n} 2^{k/2} \beta_n^k \, S_k^n \left(p_{k,n}^{\mathrm{TILT}}(\mathbf{i}, \mathbf{x}) \right) \\
&= \sum_{k=0}^{n} 2^{k/2} \beta^k \, n^{-3k/4} S_k^n \left(n^{k/2} p_{k,n}^{\mathrm{TILT}}(\mathbf{i}, \mathbf{x}) \right) \qquad (108)
\end{aligned}
$$

where the second inequality follows from (107) and the last equality follows from the linearity of S_n^k as stated in Theorem 4.2. We can now prove convergence of the $\mathfrak{Z}_n^{\mathrm{TILT}}(\beta)$.

Theorem 4.5. *Assume that the random environment has mean zero and variance one. Then as $n \to \infty$, we have*

$$
\mathfrak{Z}_n^{\mathrm{TILT}}(x, \beta) \xrightarrow{d} \mathcal{Z}_{\sqrt{2}\beta, h}, \qquad (109)
$$

where $\mathcal{Z}_{\beta,h}$ was defined in Theorem 2.1.

Consider $\boldsymbol{\rho}^{\mathrm{TILT}}(\beta) = (1, \beta\rho_1, \beta^2\rho_2, \dots) \in \mathcal{F}(\mathcal{H})$ (recall the definition of $\mathcal{F}(\mathcal{H})$ from (79)). Using the calculations from (88) to (89), together with the isometry $I : \mathcal{F}(\mathcal{H}) \to L^2(\Omega)$, we see that $\boldsymbol{\rho}^{\mathrm{TILT}}(\beta)$ is a well-defined element of $\mathcal{F}(\mathcal{H})$. **NOTE:** This sentence needs to be removed and changes when I add the Fock Space Section. Applying Theorem 4.3, we obtain that for all $\beta > 0$,

$$
\sum_{k=0}^{n} \beta^k n^{-3k/4} S_k^n(\rho_k^{\mathrm{TILT}}) \xrightarrow{(d)} \sum_{k=0}^{\infty} \beta^k \int_{[0,1]^k} \int_{\mathbb{R}^k} \rho_k^{\mathrm{TILT}}(\mathbf{t}, \mathbf{x}) \, W^{\otimes k}(d\mathbf{t}, d\mathbf{x}) \qquad (110)
$$

Let us consider the $L^2(\Omega, \mathcal{F}, Q)$ difference between (108) and

$$
\sum_{k=0}^{\infty} 2^{k/2} \beta^k n^{-3k/4} S_k^n \left(n^{k/2} \rho_k^{\mathrm{TILT}}(\boldsymbol{i}, \boldsymbol{x}) \right). \qquad (111)
$$

The difference between (108) and (111) is given by

$$
\left\| \sum_{k=0}^{\infty} 2^{k/2} \beta^k n^{-3k/4} S_k^n \left(n^{k/2} p_{k,n}^{\mathrm{TILT}}(\boldsymbol{i}, \boldsymbol{x}) \right) - \sum_{k=0}^{n} 2^{k/2} \beta^k n^{-3k/4} S_k^n \left(\rho_k^{\mathrm{TILT}}(\boldsymbol{i}, \boldsymbol{x}) \right) \right\|_{L^2(\Omega, \mathcal{F}, Q)} \qquad (112)
$$

$$
\leq \left\| \sum_{k=0}^{n} 2^{k/2} \beta^k n^{-3k/4} \left[S_k^n \left(n^{k/2} p_{k,n}^{\mathrm{TILT}}(\boldsymbol{i}, \boldsymbol{x}) \right) - S_k^n \left(\rho_k^{\mathrm{TILT}}(\boldsymbol{i}, \boldsymbol{x}) \right) \right] \right\|_{L^2(\Omega, \mathcal{F}, Q)} \qquad (113)
$$

$$
+ \left\| \sum_{k=n+1}^{\infty} 2^{k/2} \beta^k n^{-3k/4} S_k^n \left(n^{k/2} p_{k,n}^{\mathrm{TILT}}(\boldsymbol{i}, \boldsymbol{x}) \right) \right\|_{L^2(\Omega, \mathcal{F}, Q)}. \qquad (114)
$$

Line (113) can be bounded above by an application of the triangle inequality.

$$\begin{aligned}
&\left\| \sum_{k=0}^{n} 2^{k/2}\beta^k n^{-3k/4} \left(S_k^n \left(\rho_k^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x})\right) - S_k^n \left(n^{k/2} p_{k,n}^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x}) \right) \right) \right\|_{L^2(\Omega,\mathcal{F},Q)} \\
&\le \sum_{k=0}^{n} 2^{k/2}\beta^k n^{-3k/4} \left\| S_k^n \left(\rho_k^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x}) - n^{k/2} p_{k,n}^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x}) \right) \right\|_{L^2(\Omega,\mathcal{F},Q)} \\
&\le \sum_{k=0}^{n} 2^{k/2}\beta^k \left\| \rho_{k,n}^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x}) - n^{k/2} \rho_k^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x}) \right\|_{L^2([0,1]^k\times\mathbb{R}^k)}. \qquad (115)
\end{aligned}$$

where the last line follows from an application of Theorem 4.2. Note that by the local limit theorem Theorem A.1, we have

$$p_k^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x}) - n^{k/2} p_{k,n}(\boldsymbol{i},\boldsymbol{x}) \to 0 \qquad \text{pointwise as } n \to \infty.$$

Therefore, if one can provide a uniform bound on $n^{k/2} p_k^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x})$, then an application of dominated convergence shows that (115) tends to zero as $n \to \infty$.

It is important to remember that from the definition of (105) that $p_{k,n}(\boldsymbol{i},\boldsymbol{x})$ is designed to be zero for $k > n$. Therefore, we may take the limit inside the sum in (115) without worrying about the indexing of the sum, once we have the appropriate uniform bounds. We wish to estimate is the following $L^2([0,1]^k \times \mathbb{R}^k)$ norm:

$$\sup_{n\in\mathbb{N}} \left\| n^{k/2} p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x}) \right\|_{L^2([0,1]^k\times\mathbb{R}^k)}^2 \qquad (116)$$

We note that from the definition of $p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x})$ that

$$\begin{aligned}
\left\| n^{k/2} p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x}) \right\|_{L^2([0,1]^k\times\mathbb{R}^k)}^2 &= n^k \sum_{\mathbf{i}\in D_k^n} \sum_{\mathbf{x}\in\mathbb{Z}^k} \left(p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x}) \right)^2 n^{-3k/2} \\
&= n^{-k/2} \sum_{\mathbf{i}\in D_k^n} \sum_{\mathbf{x}\in\mathbb{Z}^k} \left(p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x}) \right)^2 \\
&\le n^{-k/2} \sum_{\mathbf{i}\in D_k^n} \max_{\mathbf{x}\in\mathbb{Z}^k} p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x}) \sum_{\mathbf{x}\in\mathbb{Z}^k} p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x}).
\end{aligned} \qquad (117)$$

where the first line is justified by noting that, under the diffusive scaling in the definition of (105) that produces a factor of $n^{-3k/2}$ and that the kernel $p_{k,n}^{\mathrm{TILT}}$ has already been averaged over the rectangles in (92), and is therefore constant on each such rectangle. Consequently, the L^2 norm reduces to a sum over the spatial lattice and the index set D_n^k. For each fixed $\mathbf{i}$, we can apply the law of total probability to the final sum in (117), which leads to,

$$
\begin{aligned}
n^{-k/2}\sum_{\mathbf{i}\in D_k^n}\max_{\mathbf{y}\in\mathbb{Z}^k}\sum_{\mathbf{x}\in\mathbb{Z}^k} p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x}) &= n^{-k/2}\sum_{\mathbf{i}\in D_k^n}\max_{\mathbf{y}\in\mathbb{Z}^k} p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{y})\\
&\le c^k n^{-k/2}\sum_{\mathbf{i}\in D_k^n}\prod_{j=1}^k \frac{1}{\sqrt{\mathbf{i}_j-\mathbf{i}_{j-1}}}\\
&\le c^k \int_{\Delta_k}\prod_{j=1}^k \frac{1}{\sqrt{t_j-t_{j-1}}}\,d\mathbf{t}\\
&\le c^k \left\| p_k^{\mathrm{TILT}}(\mathbf{i},\mathbf{x})\right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)} \qquad (118)
\end{aligned}
$$

Where the first inequality follows Remark 3, the second follows from an approximation of the sum to the Riemann integral and the last inequalities follow from the Drichlet type intergral in Subsection E.4. The constant c is a constant which changes line by line. Taking the supremum over in n in (118), we can produce the uniform bound that is needed for (116)

$$
\sup_{n\in\mathbb{N}}\left\| n^{k/2} p_{k,h}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x})\right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)} \le c^k \left\| p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x})\right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)}. \qquad (119)
$$

justifying the application of the dominated convergence theorem in (115). For the term (114), an application of the Triangle inequality shows that

$$
\begin{aligned}
&\left\|\sum_{k=n+1}^{\infty} 2^{k/2}\beta^{2k} n^{-3k/2} S_k^n\left(p_k^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x})\right)\right\|^2_{L^2(\mathcal{F},\Omega,Q)}\\
&\le \sum_{k=n+1}^{\infty}\left\| 2^{k/2}\beta^{2k} n^{-3k/2} S_k^n\left(p_k^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x})\right)\right\|^2_{L^2(\mathcal{F},\Omega,Q)}\\
&\le \sum_{k=n+1}^{\infty} 2^k\beta^k n^{-3k/4}\left\|\sum_{\boldsymbol{i}\in E_n^k}\sum_{\boldsymbol{x}\in\mathbb{Z}^k}\overline{p_k^{\mathrm{TILT}}}\left(\frac{\boldsymbol{i}}{n},\frac{\boldsymbol{x}}{\sqrt{n}}\right)\mathbb{1}_{\{\boldsymbol{i}\leftrightarrow\boldsymbol{x}\}}\,\omega(\boldsymbol{i},\boldsymbol{x})\right\|^2_{L^2(\mathcal{F},\Omega,Q)}.
\end{aligned}
\qquad (120)
$$

To bound (120), we consider the covariance structure for $k_1\neq k_2$, and $\mathbf{i}\in E_{k_1}^n$, $\mathbf{x}\in\mathbb{Z}^{k_1}$ $\mathbf{i}'\in E_{k_2}^n$, $\mathbf{x}\in\mathbb{Z}^{k_2}$. We note that

$$
Q\left[\prod_{j=1}^{k_1}\omega(\boldsymbol{i}_{j_1},\boldsymbol{x}_{j_1}),\ \prod_{j=1}^{k_2}\omega(\boldsymbol{i}_{j_2},\boldsymbol{x}_{j_2})\right]=0 \qquad (121)
$$

since there is at least one ω which is distinct from the rest, and the independence of each $\omega(i,x)$ implies that (121) must equal zero when $k_1 \neq k_2$. However if $k_1 = k_2$, then

$$Q\left[\prod_{j=1}^{k_1} \omega(\boldsymbol{i}_{j_1}, \boldsymbol{x}_{j_1}),\ \prod_{j=1}^{k_2} \omega(\boldsymbol{i}_{j_2}, \boldsymbol{x}_{j_2})\right] = \mathbb{1}_{\{\mathbf{i}=\mathbf{i}',\mathbf{x}=\mathbf{x}'\}} \tag{122}$$

Consequently, returning to (120), we can calculate the norm in this case and this produces,

$$\sum_{k=n+1}^{\infty} 2^k \beta^{2k} n^{-3k/2} \sum_{\boldsymbol{i}\in E_n^k} \sum_{\boldsymbol{x}\in\mathbb{Z}^k} \left(\overline{p_k^{\mathrm{TILT}}}\left(\frac{\boldsymbol{i}}{n}, \frac{\boldsymbol{x}}{\sqrt{n}}\right) \mathbb{1}_{\{\boldsymbol{i}\leftrightarrow\boldsymbol{x}\}}\right)^2 \tag{123}$$

$$\leq \sum_{k=n+1}^{\infty} 2^k \beta^{2k} n^{-3k/2} \sum_{\boldsymbol{i}\in E_n^k} \sum_{\boldsymbol{x}\in\mathbb{Z}^k} \frac{1}{|R|} \int_R \left(p_k^{\mathrm{TILT}}(\boldsymbol{y},\boldsymbol{z})\right)^2 d\boldsymbol{z}\, d\boldsymbol{y} \tag{124}$$

$$\leq \sum_{k=n+1}^{\infty} 2^k \beta^{2k} n^{-3k/2} \sum_{\boldsymbol{i}\in [n]^k} \sum_{\boldsymbol{x}\in\mathbb{Z}^k} \frac{1}{|R|} \int_R \left(p_k^{\mathrm{TILT}}(\boldsymbol{y},\boldsymbol{z})\right)^2 d\boldsymbol{z}\, d\boldsymbol{y} \tag{125}$$

$$= \sum_{k=n+1}^{\infty} 2^k \beta^{2k} \int_{[0,1]^k} \int_{\mathbb{R}^k} \left(p_k^{\mathrm{TILT}}(\boldsymbol{y},\boldsymbol{z})\right)^2 d\boldsymbol{z}\, d\boldsymbol{y} \tag{126}$$

$$= \sum_{k=n+1}^{\infty} 2^k \beta^{2k} \left\|p_k^{\mathrm{TILT}}\right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)}. \tag{127}$$

where from (123) to (124) we have used the definition of (92) plus and application of Jensen's inequality. To go from line (124) to (125) we have use the set containment **NOTE:** this seems wrong

$$\{\mathbf{i} \in E_k^n\} \subseteq \{\mathbf{i} \in [n]^k\} \tag{128}$$

To justify (125) to (126) recall that from that our rectangles are dependent on both $\mathbf{x}$ and $\mathbf{i}$, along with recalling that the size of each rectangle is $2^k n^{-\frac{3k}{2}}$, which provides a Riemann approximation of the sum to the intergral. Therefore we have a $L^2(\mathcal{F},\Omega,Q)$ to $L^2([0,1]^k\times\mathbb{R}^k)$ bound for (114) which is given by

$$\left\|\sum_{k=n+1}^{\infty} 2^{k/2}\beta^k n^{-3k/4} S_k^n\left(p_k^{\mathrm{TILT}}(\boldsymbol{i},\boldsymbol{x})\right)\right\|_{L^2(\mathcal{F},\Omega,Q)} \leq \sum_{k=n+1}^{\infty} 2^k \beta^k \left\|p_k^{\mathrm{TILT}}\right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)} \tag{129}$$

In Subsection E.4 we show that $\left\|p_k^{\mathrm{TILT}}\right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)} = 1/2^k\,\Gamma\left(\left(\frac{k}{2}\right)+1\right)$. Therefore taking the limit as $n\to\infty$ shows that the term in (129) tends to zero.

Since both (113) and (114) tend to zero as we take $n \to \infty$, the difference term (144) also goes to zero and hence we get the required result.

Perturbation Of The Random Environment We are now able to prove convergence in law of $Z_n^{\mathrm{TILT}}(\beta)$ by considering a change in environment.

Theorem 4.6. *Assume that there exists a $\beta > 0$ such that*

$$\lambda(\beta) := \log Q(e^{\beta\omega}) < \infty. \tag{130}$$

NOTE: *notation here doesnt take into account scaling of h Then the renormalised partition function*

$$e^{-n\lambda(\beta n^{-1/4})} Z_n^{\mathrm{TILT}}(\beta n^{-1/4}) \tag{131}$$

converges in distribution to $\mathcal{Z}_{\sqrt{2}\beta,h}$ as defined in Theorem 2.1.

Proof. Set $\beta_n := \beta n^{-1/4}$. Define a new environment $\tilde{\omega}$ by

$$e^{\beta_n \omega(i,x) - \lambda(\beta_n)} = 1 + \beta_n \tilde{\omega}(i,x). \tag{132}$$

This allows us to write

$$\begin{aligned} e^{-n\lambda(\beta_n)} Z_n^{\mathrm{TILT}}(\beta n^{-1/4}) &= P_h^{\mathrm{TILT}}\left[\prod_{i=1}^{n} e^{\beta_n \omega(i,S_i) - \lambda(\beta_n)}\right] \\ &= P_h^{\mathrm{TILT}}\left[\prod_{i=1}^{n} (1 + \beta_n \tilde{\omega}(i,S_i))\right] := \mathfrak{Z}_n^{\mathrm{TILT}}(\tilde{\omega}_n, \beta). \end{aligned} \tag{133}$$

We have introduced the notation $\mathfrak{Z}_n^{\mathrm{TILT}}(\tilde{\omega}_n, \beta)$ to clarify the change in environment. One can check that the right-hand side of (256) has expectation one, and consequently $Q[\tilde{\omega}(i,x)] = 0$. The variance computation follows from a standard second moment computation; see [Com17]. Defining $\lambda_2(\beta_n) := \lambda(2\beta_n) - 2\lambda(\beta_n)$, one obtains $\lambda_2(\beta_n) = \frac{\beta_n^2}{2} + O(\beta_n^3)$, which implies $Q[\tilde{\omega}^2(i,x)] = 1 + O(\beta_n)$.

The conditions in Theorem 4.4 can be easily checked and hence as a result,

$$\sum_{k=0}^{n} \beta^k n^{-3k/4}\, \mathbf{S}_k^n(p_k^{\mathrm{TILT}}; \tilde{\omega}_n) \xrightarrow{(d)} \sum_{k=0}^{\infty} \beta^k \int_{[0,1]^k} \int_{\mathbb{R}^k} p_k^{\mathrm{TILT}}(\mathbf{t}, \mathbf{x})\, W^{\otimes k}(d\mathbf{t}, d\mathbf{x}) \;=:\; \mathcal{Z}_{\beta,h}. \tag{134}$$

as $n \to \infty$. Given that the environment has variance $1 + O(\beta_n)$, then condition (100) can be readily checked and implies that

$$\left\| \sum_{k=0}^{n} 2^{k/2}\beta^k n^{-3k/4} \left(S_k^n \left(\rho_{k,h}^n(\boldsymbol{i}, \boldsymbol{x}); \tilde{\omega}_n \right) - S_k^n \left(n^{k/2} p_{k,n}^{\mathrm{TILT}}(\boldsymbol{i}, \boldsymbol{x}); \tilde{\omega}_n \right) \right) \right\|_{L^2(\mathcal{F},\Omega,Q)} \tag{135}$$

goes to zero, which follows from an application of the same inequalities given in Theorem 4.5. Furthermore, the term

$$\left\| \sum_{k=n+1}^{\infty} 2^{k/2}\beta^k n^{-3k/4} S_k^n \left(\rho_{k,h}^n(\boldsymbol{i}, \boldsymbol{x}; \tilde{\omega}(\boldsymbol{i}, \boldsymbol{x})) \right) \right\|_{L^2(\mathcal{F},\Omega,Q)} \tag{136}$$

also goes to zero as $n \to \infty$ for the same reasons also given in Theorem 4.5 and Remark 3. $\square$

The Point-to-Line Partition Function In this section, we will prove the convergence of the modified tilted point-to-point partition function

$$\mathfrak{Z}_n^{\mathrm{TILT}}(x,\beta) = E^{\mathrm{TILT}}\left[\prod_{i=1}^{n} (1 + \beta\omega_{i,S_i})\, \mathbb{1}_{\{S_n = x\}} \right]. \tag{137}$$

The proof of the point-to-line case is much the same as the proof of the modified tilted partition function given in Theorem 4.5, with some key differences. Since we are fixing the endpoint under diffusive scaling, we need to introduce the analogue of (105). Define $p_{k|x,n}^{\mathrm{TILT}}$ on $[0,1]^k \times \mathbb{R}^k$ by

$$p_{k|x,n}^{\mathrm{TILT}}(\mathbf{t}, \mathbf{x}) = P(X_{t_1} \in dx_1, \ldots, X_{t_k} \in dx_k \,|\, X_1 \in dx)\, \mathbb{1}_{\{\lceil n\mathbf{t} \rceil \in D_n^k\}}. \tag{138}$$

NOTE: Do you exactly know what this means Roan? This allows us to write the analogous U-statistic and proceed with the proof of Theorem 4.5. To upgrade our full result to $Z_n^{\mathrm{TILT}}(x,\beta)$, we will use the same change in environment given in (131) and provide estimates so that the analogues of Theorem 4.4 remain satisfied. We can also describe the limiting density of (138), whose k-fold transition probabilities are given by

$$\rho_{k|x}^{\mathrm{TILT}}(\mathbf{t}, \mathbf{x}) = \frac{\rho^{\mathrm{TILT}}(1 - t_k,\, x - x_k)\, \rho_k^{\mathrm{TILT}}(\mathbf{t}, \mathbf{x})}{\rho^{\mathrm{TILT}}(1, x)} \tag{139}$$

One can also check that the random variable

$$X = \sum_{k=0}^{\infty} W^{\otimes k}\left(\rho_{k|x}^{\mathrm{TILT}} \right) \tag{140}$$

is a well-defined element of the Fock space. Indeed, by a calculation very similar to (89), we obtain

$$\int_{\Delta_k}\int_{\mathbb{R}^k}\left(\rho^{\mathrm{TILT}}_{k|x}(\mathbf{t},\mathbf{x})\right)^2 d\mathbf{x}\,d\mathbf{t} = \int_{\Delta_k}\prod_{i=1}^{k-1}\frac{1}{t_i - t_{i-1}\sqrt{1-t_k}}\,\frac{e^{-x^2}}{\rho^{\mathrm{TILT}}(1,x)}\,d\mathbf{t}. \tag{141}$$

where the factor of e^{-x^2} comes from completing the square in the exponential in the last spatial integral. One can check that (141) is a Dirichlet integral, and in Subsection E.4, we show integrals of this form can be written as the ratio of gamma functions, whose ratios decay superexponentially. In the same spirit of (108) we can write

$$\begin{aligned}
\mathfrak{Z}^{\mathrm{TILT}}_n(\beta_n, x) &= \sum_{k=0}^{n}\beta_n^k \sum_{\mathbf{i}\in D_k^n}\sum_{\mathbf{x}\in\mathbb{Z}^k}\left(\prod_{j=1}^{k}\omega(i_j,x_j)p^{\mathrm{TILT}}_{k|x}(i_j - i_{j-1},\, x_j - x_{j-1})\right)\\
&= \sum_{k=0}^{n} 2^{k/2}\beta_n^k\, S_k^n\left(p^{\mathrm{TILT}}_{k|x,n}(\mathbf{i},\mathbf{x})\right)\\
&= \sum_{k=0}^{n} 2^{k/2}\beta^k\, n^{-3k/4} S_k^n\left(n^{k/2}p^{\mathrm{TILT}}_{k|x,n}(\mathbf{i},\mathbf{x})\right)
\end{aligned}\tag{142}$$

Applying Theorem 4.3, we obtain convergence of finite-dimensional distributions: for all $\beta > 0$,

$$\left\{x \longmapsto \sum_{k=0}^{n}\beta^k n^{-3k/4} S^n_{k|x}\left(\rho^{\mathrm{TILT}}_{k|x}\right)\right\} \xrightarrow{(d)} \left\{x \longmapsto \sum_{k=0}^{\infty}\beta^k\int_{[0,1]^k}\int_{\mathbb{R}^k}\rho^{\mathrm{TILT}}_{k|x}(\mathbf{t},\mathbf{y})\,W^{\otimes k}(d\mathbf{t},d\mathbf{y})\right\}. \tag{143}$$

As in the proof of Theorem 4.5 , we would like to estimate the L^2 difference between The difference between (142) and the RHS of (143) is given by

$$\left\|\sum_{k=0}^{\infty}2^{k/2}\beta^k n^{-3k/4}S_k^n\left(n^{k/2}p^{\mathrm{TILT}}_{k|x,n}(\boldsymbol{i},\boldsymbol{x})\right) - \sum_{k=0}^{n}2^{k/2}\beta^k n^{-3k/4}S_k^n\left(\rho^{\mathrm{TILT}}_{k|x}(\boldsymbol{i},\boldsymbol{x})\right)\right\|_{L^2(\Omega,\mathcal{F},Q)} \tag{144}$$

$$\leq \left\|\sum_{k=0}^{n}2^{k/2}\beta^k n^{-3k/4}\left[S_k^n\left(n^{k/2}p^{\mathrm{TILT}}_{k|x,n}(\boldsymbol{i},\boldsymbol{x})\right) - S_k^n\left(\rho^{\mathrm{TILT}}_{k|x}(\boldsymbol{i},\boldsymbol{x})\right)\right]\right\|_{L^2(\Omega,\mathcal{F},Q)} \tag{145}$$

$$+\left\|\sum_{k=n+1}^{\infty}2^{k/2}\beta^k n^{-3k/4}S^n_{k|x}\left(n^{k/2}p^{\mathrm{TILT}}_{k|x,n}(\boldsymbol{i},\boldsymbol{x})\right)\right\|_{L^2(\Omega,\mathcal{F},Q)}. \tag{146}$$

For (146), we can repeat the calculations given in the proof of Theorem 4.5 (see the lines starting just above (120) through line (129)) to conclude that

$$\left\| \sum_{k=n+1}^{\infty} 2^{k/2}\beta^k n^{-3k/4} S^n_{k|x}\left(n^{k/2} p^{\mathrm{TILT}}_{k|x,n}(\boldsymbol{i},\boldsymbol{x})\right)\right\|_{L^2(\Omega,\mathcal{F},Q)} \leq \sum_{k=n+1}^{\infty} 2^k\beta^k \left\|p^{\mathrm{TILT}}_k\right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)}. \tag{147}$$

which, by (141), decays uniformly in x as $n\to\infty$. For term (146), an application of the triangle inequality plus the use of the estimates in Theorem 4.2 produces the bound

$$\begin{aligned} &\left\| \sum_{k=n+1}^{\infty} 2^{k/2}\beta^k n^{-3k/4} S^n_{k|x}\left(n^{k/2} p^{\mathrm{TILT}}_{k|x,n}(\boldsymbol{i},\boldsymbol{x})\right)\right\|_{L^2(\Omega,\mathcal{F},Q)} \\ &\leq \sum_{k=0}^{n} 2^{k/2}\beta^k \left\|\rho^{\mathrm{TILT}}_{k|x,n}(\boldsymbol{i},\boldsymbol{x}) - n^{k/2}p^{\mathrm{TILT}}_{k|x}(\boldsymbol{i},\boldsymbol{x})\right\|_{L^2([0,1]^k\times\mathbb{R}^k)} \end{aligned} \tag{148}$$

NOTE: you should probably do this calculation. One can provide uniform bounds on $n^{k/2}P^{\mathrm{TILT}}_{k|x}$ in analogy with lines (116) and (118), and an application of the dominated convergence theorem plus Theorem A.1 shows that (148) goes to zero as $n\to\infty$. We can upgrade our results to convergence of the full partition function $Z^{\mathrm{TILT}}_n(x.\beta)$ following the change in environment given by (255) and an application of Theorem 4.4.

The Five-Parameter Process We are now in a position to prove the convergence of the five-parameter process $\mathfrak{Z}^{\mathrm{TILT}}_n(s,y;x,t,\beta)$. As before, we define $p^{\mathrm{TILT}}_{k|(s,y;x,n)}$ on $[0,1]^k\times\mathbb{R}^k$ by

$$p^{\mathrm{TILT}}_{k|(s,y;x,n)}(\mathbf{t},\mathbf{x}) = n^{k/2}P(X_{t_1}\in dx_1,\ldots,X_{t_k}\in dx_k \mid X_t\in dx,\ X_s\in dy)\,\mathbb{1}_{\{\lceil n\mathbf{t}\rceil\in D^k_n\}}. \tag{149}$$

These kernels are space-time shifts of the kernels

$$n^{k/2}P(X_{t_1}\in dx_1,\ldots,X_{t_k}\in dx_k \mid X_t\in dx)\,\mathbb{1}_{\{\lceil n\mathbf{t}\rceil\in D^k_n\}}.$$

Since the underlying simple random walk is shift-invariant, the shifted kernel has the same law as $p^{\mathrm{TILT}}_{k|(s,y;x,n)}$, with a shifted environment. Using the Markov property, one checks that the distribution is shift-invariant (see [Com17]). Hence we obtain the following equality in distribution:

$$\frac{\sqrt{n}}{2}\,\mathfrak{Z}^{\mathrm{TILT}}_n(sn, y\sqrt{n}; x, tn, \beta n^{-1/4}) \overset{d}{=} \frac{\sqrt{n}}{2}\,\mathfrak{Z}^{\mathrm{TILT}}_n\left(0,0;\,(x-y)\sqrt{n},\,n(t-s),\,\beta n^{-1/4}\right). \tag{150}$$

Hence, the convergence of finite-dimensional distributions follows from Section 4.

Identification of the Chaos Expansion and the PAM with Drift. The chaos expansion presented in **??** can be identified as the unique solution to the Parabolic Anderson Model (PAM) with drift by extending the framework of [BC95]. As defined in (86), the kernel ρ^{TILT} corresponds to the transition probabilities of a Brownian motion with a constant drift parameter h. Since the Lebesgue measure is translation-invariant, the L^2 norm of the drifted kernel ρ^{TILT} is identical to that of the standard heat kernel, and thus remains finite (see e.g., [**?**]). Consequently, the integrability conditions required for the convergence of the Wiener chaos series and the pathwise uniqueness of the solution are satisfied. It follows from [BC95, Theorem 2.2 (ii)] that this expansion is the unique solution to the PAM with drift if the condition is checked in **??**. We need to check that

$$\sup_{t\in(0,T]}\sup_{x\in\mathbb{R}}\int_0^t\int_0^s\int_{\mathbb{R}}\int_{\mathbb{R}}\rho^{\mathbf{TILT}}(t-s,x-y)^2\,\rho^{\mathbf{TILT}}(s-s',y-y')^2\,\mathbb{E}\big[\mathbb{Z}_{\sqrt{2\beta}}(y')^2\big]\;dy'\,dy\,ds'\,ds<\infty. \tag{151}$$

It follows from the calculation in (**??**) that

$$\mathbb{E}\Big[\mathcal{Z}_{\sqrt{2}\beta}(y',s')^2\Big]\le C(\beta)\,\rho_h(x,t;y',s'). \tag{152}$$

Consequently, using the semigroup property of the shifted heat kernels, we obtain

$$\begin{aligned}
&\int_0^t\int_0^s\int_{\mathbb{R}}\int_{\mathbb{R}}\rho^{\mathrm{TILT}}(t-s,x-y)^2\,\rho^{\mathrm{TILT}}(s-s',y-y')^2\,\mathbb{E}\Big[\mathcal{Z}_{\sqrt{2}\beta}(y',s')^2\Big]\;dy'\,dy\,ds'\,ds\\
&\quad\le C(\beta)\int_0^t\int_0^s\int_{\mathbb{R}}\int_{\mathbb{R}}\rho^{\mathrm{TILT}}(t-s,x-y)^2\,\rho^{\mathrm{TILT}}(s-s',y-y')^2\,\rho_h(y',s')\,dy'\,dy\,ds'\,ds\\
&\quad\le C(\beta)\int_0^t\int_0^s e^{-x^2}\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s-s'}}\frac{1}{\sqrt{s'}}\,ds'\,ds.
\end{aligned} \tag{153}$$

The final expression can be identified as a Dirichlet-type integral, see Subsection E.4, and one can check that

$$\le C(\beta)\int_0^t\int_0^s e^{-x^2}\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s-s'}}\frac{1}{\sqrt{s'}}\,ds'\,ds\le C(\beta)\sqrt{t}e^{-x^2} \tag{154}$$

taking the supremum over space and time over the compact interval $(0,T]$ produces the result.

Going to the SPDE in h and t **NOTE:** This is most likely going to be informal at this stage.

Let $\overline{W}(t,h)$ denote space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$, and consider the following SPDE:

$$\partial_t U(t,h) = \frac{1}{2t^2}\partial_{hh}U(t,h) + \frac{h_0 - h}{t}\partial_h U(t,h) + \frac{\beta}{t^{1/2}} U(t,h)\,\overline{W}(t,h). \tag{155}$$

Here h_0 is the original drift parameter. The deterministic part of (155) (i.e. the $\beta = 0$ case) is the Fokker–Planck operator for a Brownian bridge pinned to h_0 at time t, with diffusion coefficient $1/(2t^2)$ and bridge drift $(h_0 - h)/t$.

One can check formally that, under Dirac initial data $U(t_0, \cdot) = \delta_{h_0}(\cdot)$ at time $t_0 > 0$, the mild (Duhamel) formulation of (155) is

$$U(t,h) = K_{h_0}(t_0, h_0; t, h) + \beta \int_{t_0}^{t}\int_{\mathbb{R}} K_{h_0}(s,\eta; t,h)\,\frac{1}{s^{1/2}}\,U(s,\eta)\,\overline{W}(ds, d\eta), \tag{156}$$

where the Green's function of the deterministic part is

$$K_{h_0}(s,\eta;t,h) = \frac{t}{\sqrt{2\pi(t-s)}}\exp\left\{-\frac{t^2\left(h - \frac{s}{t}\eta - h_0\left(1-\frac{s}{t}\right)\right)^2}{2(t-s)}\right\}. \tag{157}$$

Equivalently, the expression inside the exponential simplifies as

$$t^2\left(h - \frac{s}{t}\eta - h_0\left(1 - \frac{s}{t}\right)\right)^2 = \big(th - s\eta - h_0(t-s)\big)^2, \tag{158}$$

which makes the connection to the original spatial kernel transparent.

Iterating the mild solution (166) gives the formal Wiener chaos expansion

$$\begin{aligned} U(t,h) &= K_{h_0}(t_0,h_0;t,h) \\ &\quad + \sum_{k=1}^{\infty}\beta^k \int_{\Delta_k(t_0,t)}\int_{\mathbb{R}^k} K_{h_0}(t_0,h_0;s_1,\eta_1)\prod_{i=1}^{k-1}\left[\frac{1}{s_i^{1/2}}K_{h_0}(s_i,\eta_i;s_{i+1},\eta_{i+1})\right] \\ &\quad \times \frac{1}{s_k^{1/2}}K_{h_0}(s_k,\eta_k;t,h)\,\overline{W}(ds_1,d\eta_1)\cdots\overline{W}(ds_k,d\eta_k), \end{aligned} \tag{159}$$

where

$$\Delta_k(t_0,t) = \big\{(s_1,\ldots,s_k) : t_0 < s_1 < \cdots < s_k < t\big\}.$$

We start from the usual spatial mild formulation of the stochastic heat equation with drift:

$$Z(t,x) = \rho_{h_0}(t_0,x_0;t,x) + \beta\int_{t_0}^{t}\int_{\mathbb{R}}\rho_{h_0}(s,y;t,x)Z(s,y)\,W(ds,dy), \tag{160}$$

where

$$\rho_{h_0}(s,y;t,x) = \frac{1}{\sqrt{2\pi(t-s)}}\exp\left\{-\frac{(x-y-h_0(t-s))^2}{2(t-s)}\right\}.$$

We now introduce the slope coordinate

$$x = th, \qquad y = s\eta.$$

Accordingly, define the slope-coordinate field

$$U(t,h) := t\,Z(t,th). \tag{161}$$

The factor t appears because the change of variables $x = th$ gives

$$dx = t\,dh.$$

Thus $U(t,h)$ is the density-normalized version of $Z(t,x)$ in the h-coordinate.

Multiplying (160) by t and setting $x = th$, we obtain

$$\begin{aligned} U(t,h) &= t\,\rho_{h_0}(t_0,x_0;t,th) \\ &\quad + \beta \int_{t_0}^{t}\int_{\mathbb{R}} t\,\rho_{h_0}(s,y;t,th) Z(s,y)\,W(ds,dy). \end{aligned} \tag{162}$$

Now set

$$y = s\eta.$$

Then the drifted heat kernel becomes

$$\begin{aligned} K_{h_0}(s,\eta;t,h) &:= t\,\rho_{h_0}(s,s\eta;t,th) \\ &= \frac{t}{\sqrt{2\pi(t-s)}} \exp\left\{ -\frac{\big(th - s\eta - h_0(t-s)\big)^2}{2(t-s)} \right\}. \end{aligned} \tag{163}$$

Equivalently,

$$K_{h_0}(s,\eta;t,h) = \frac{t}{\sqrt{2\pi(t-s)}} \exp\left\{ -\frac{t^2\left(h - \frac{s}{t}\eta - h_0\left(1-\frac{s}{t}\right)\right)^2}{2(t-s)} \right\}. \tag{164}$$

Next observe that, by the definition of U,

$$U(s,\eta) = s\,Z(s,s\eta).$$

Hence

$$Z(s,s\eta) = \frac{1}{s} U(s,\eta).$$

Also, since $y = s\eta$, we have

$$dy = s\,d\eta.$$

Space-time white noise transforms according to the square-root of the Jacobian. Therefore,

$$W(ds,dy) = W(ds,s\,d\eta) = s^{1/2}\,\overline{W}(ds,d\eta),$$

where $\overline{W}$ denotes space-time white noise in the (s,η)-coordinates.

Substituting these identities into (162), we get

$$\begin{aligned}U(t,h) &= K_{h_0}(t_0,h_0;t,h)\\ &\quad + \beta \int_{t_0}^{t}\int_{\mathbb{R}} K_{h_0}(s,\eta;t,h)\frac{1}{s}U(s,\eta)s^{1/2}\,\overline{W}(ds,d\eta).\end{aligned} \tag{165}$$

Therefore

$$U(t,h) = K_{h_0}(t_0,h_0;t,h) + \beta \int_{t_0}^{t}\int_{\mathbb{R}} K_{h_0}(s,\eta;t,h)\frac{1}{s^{1/2}}U(s,\eta)\,\overline{W}(ds,d\eta). \tag{166}$$

Thus the factor $s^{-1/2}$ is not inserted by hand. It comes from combining

$$Z(s,s\eta) = \frac{1}{s}U(s,\eta)$$

with the white-noise scaling

$$W(ds,s\,d\eta) = s^{1/2}\,\overline{W}(ds,d\eta).$$

In other words,

$$\frac{1}{s}\times s^{1/2} = s^{-1/2}.$$

Iterating (166) gives the formal Wiener chaos expansion

$$\begin{aligned}U(t,h) &= K_{h_0}(t_0,h_0;t,h)\\ &\quad + \sum_{k=1}^{\infty}\beta^k \int_{\Delta_k(t_0,t)}\int_{\mathbb{R}^k} K_{h_0}(t_0,h_0;s_1,\eta_1)\\ &\qquad \times \prod_{r=1}^{k-1}\left[\frac{1}{s_r^{1/2}}K_{h_0}(s_r,\eta_r;s_{r+1},\eta_{r+1})\right]\\ &\qquad \times \frac{1}{s_k^{1/2}}K_{h_0}(s_k,\eta_k;t,h)\,\overline{W}(ds_1,d\eta_1)\cdots\overline{W}(ds_k,d\eta_k),\end{aligned} \tag{167}$$

where

$$\Delta_k(t_0,t) = \{(s_1,\ldots,s_k) : t_0 < s_1 < \cdots < s_k < t\}.$$

Is this an element of the FOck Space? For fixed k, define

$$F_k(\mathbf{s},\boldsymbol{\eta};t,h) = K_{h_0}(t_0,h_0;s_1,\eta_1)\prod_{i=1}^{k-1}\left[s_i^{1/2}K_{h_0}(s_i,\eta_i;s_{i+1},\eta_{i+1})\right]s_k^{1/2}K_{h_0}(s_k,\eta_k;t,h).$$

We need to check that

$$F_k \in L^2(\Delta_k(t_0,t)\times\mathbb{R}^k).$$

Using the Gaussian square identity, each squared transition kernel is bounded, after integrating over the spatial variables, by a factor of order

$$(s_{i+1} - s_i)^{-1/2}.$$

Hence

$$\int_{\mathbb{R}^k} |F_k(\mathbf{s}, \boldsymbol{\eta}; t, h)|^2 \, d\boldsymbol{\eta} \lesssim C^k \prod_{i=1}^{k} s_i \prod_{i=0}^{k} (s_{i+1} - s_i)^{-1/2},$$

where $s_0 = t_0$ **and** $s_{k+1} = t$**. Since** $t_0 > 0$ **and** $s_i \leq t$**, we have**

$$\prod_{i=1}^{k} s_i \leq t^k.$$

Therefore

$$\|F_k\|^2_{L^2(\Delta_k(t_0,t)\times\mathbb{R}^k)} \lesssim C^k t^k \int_{\Delta_k(t_0,t)} \prod_{i=0}^{k} (s_{i+1} - s_i)^{-1/2} \, d\mathbf{s}.$$

The simplex integral is finite and is given by

$$\int_{\Delta_k(t_0,t)} \prod_{i=0}^{k} (s_{i+1} - s_i)^{-1/2} \, d\mathbf{s} = (t - t_0)^{\frac{k-1}{2}} \frac{\Gamma(1/2)^{k+1}}{\Gamma((k+1)/2)}.$$

Thus $F_k \in L^2(\Delta_k(t_0, t) \times \mathbb{R}^k)$**, and so the** k**-fold integrand defines a valid** k**-th Wiener chaos element.**

Deriving the h-chaos expansion from the PAM with drift We now show how (263) arises from the chaos expansion for the parabolic Anderson model (PAM) with drift via a change of variables. Recall the drifted five-parameter kernel

$$\rho^{\mathrm{TILT}}_{h_0}(x, t; y, s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{\big(x - y - h_0(t-s)\big)^2}{2(t-s)} \right). \tag{168}$$

Consider the change of variables

$$x = th, \qquad y = s\eta. \tag{169}$$

Substituting (169) into (168) gives

$$\begin{aligned}
\rho^{\mathrm{TILT}}_{h_0}(th, t; s\eta, s) &= \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{\big(th - s\eta - h_0(t-s)\big)^2}{2(t-s)} \right) \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{ -\frac{t^2\left(h - \frac{s}{t}\eta - h_0\big(1 - \frac{s}{t}\big)\right)^2}{2(t-s)} \right\} \\
&= \frac{1}{t} K_{h_0}(s, \eta; t, h).
\end{aligned} \tag{170}$$

Hence $\rho_{h_0}^{\mathrm{TILT}} = \frac{1}{t} K_{h_0}$ under the substitution (169). When applying this change of variables to a spatial integral, each $\int dx$ becomes $t \int dh$ and each $\int dy$ becomes $s \int d\eta$, producing Jacobian factors of t and s respectively. In the PAM chaos expansion a generic k-th order term contains k spatial integrations $\int dy_i = s_i \int d\eta_i$ and uses the kernel relation $\rho_{h_0}^{\mathrm{TILT}} = \frac{1}{s_i} K_{h_0}$ at each intermediate time s_i. These two contributions combine as $s_i \times \frac{1}{s_i} = 1$ at every interior vertex, except that the noise coefficient in the original PAM carries an additional factor which, after substitution, leaves a $s_i^{-1/2}$ weight at each integration point. This is precisely the factor $s_i^{-1/2}$ appearing in (263).

Lemma 4.7 (Tilted local limit theorem in slope variables)**.** *Let $0 < s < t \le T$, and let $\eta, h \in \mathbb{R}$. Set*

$$i_n = \lfloor nt \rfloor, \qquad j_n = \lfloor ns \rfloor.$$

Suppose that under $\mathbb{P}_{h_0,n}^{\mathrm{TILT}}$ the random walk has drift chosen so that, under diffusive scaling,

$$\frac{S_{\lfloor nr \rfloor}}{\sqrt{n}} \Longrightarrow B_r + h_0 r.$$

Then, uniformly for $(s, \eta; t, h)$ in compact subsets of

$$\{0 < s < t \le T,\ \eta, h \in \mathbb{R}\},$$

we have

$$\sqrt{n}\, \mathbb{P}_{h_0,n}^{\mathrm{TILT}} \left(S_{i_n} - S_{j_n} = \left\lfloor \sqrt{n}\,(th - s\eta) \right\rfloor \right) \longrightarrow \rho_{h_0}^{\mathrm{TILT}}(t-s, th - s\eta),$$

where

$$\rho_{h_0}^{\mathrm{TILT}}(t-s, x) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(x - h_0(t-s))^2}{2(t-s)} \right).$$

Equivalently,

$$\sqrt{n}\, \mathbb{P}_{h_0,n}^{\mathrm{TILT}} \left(S_{i_n} - S_{j_n} = \left\lfloor \sqrt{n}\,(th - s\eta) \right\rfloor \right) \longrightarrow \frac{1}{t} K_{h_0}(s, \eta; t, h),$$

where

$$K_{h_0}(s, \eta; t, h) = t\, \rho_{h_0}^{\mathrm{TILT}}(t-s, th - s\eta).$$

That is,

$$K_{h_0}(s, \eta; t, h) = \frac{t}{\sqrt{2\pi(t-s)}} \exp\left\{ -\frac{t^2 \left(h - \frac{s}{t}\eta - h_0 \left(1 - \frac{s}{t}\right)\right)^2}{2(t-s)} \right\}.$$

The centred change of variables An alternative substitution,

$$x = t(h + h_0), \qquad y = s(\eta + h_0), \tag{171}$$

removes h_0 from the kernel entirely. Indeed,

$$x - y - h_0(t-s) = t(h + h_0) - s(\eta + h_0) - h_0(t-s) = th - s\eta, \tag{172}$$

so that

$$\rho_{h_0}^{\mathrm{TILT}}\big(t(h+h_0), t;\, s(\eta+h_0), s\big) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(th-s\eta)^2}{2(t-s)}\right)$$
$$= \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{t^2\big(h-\frac{s}{t}\eta\big)^2}{2(t-s)}\right\}. \quad (173)$$

This is the drift-free ($h_0 = 0$) version of the h-kernel. Thus the substitution (169) retains the h_0-dependence in the kernel, while the centred substitution (171) absorbs it into the coordinate shift and yields a kernel with no explicit h_0.

5 Convergence Of The Finite Dimensional Distributions For The Velocity Partition Function

The Underlying Discrete Diffusion We now introduce the discrete process which underlies the velocity partition function. Consider a SRW S_i with incriments ξ_i which are Bernoulli random varible with mean zero and variance one. Set

$$t_i = \frac{i}{n}, \qquad H_i^n = h_0 + \frac{S_i}{\sqrt{n}\, t_i}. \quad (174)$$

We will use the notation $H_i^n(x)$ to denote that the position of the SRW S_i is at postion x. This change of variables turns the random walk into a time-inhomogeneous Markov chain. An important feature of this transformation is that the spatial discretisation is no longer uniform. Indeed, the lattice spacing at time t_i is given by

$$\Delta h_i := H_i^n(x) - H_i^n(x+1) = \frac{1}{\sqrt{n}\, t_i}$$

so the mesh becomes progressively finer as time evolves. This is in contrast to the classical Donsker scaling, where the spatial mesh is uniformly of order $n^{-1/2}$. This time-dependent geometry is a distinguishing feature of the velocity coordinates and is ultimately responsible for the time-dependent coefficients appearing in both the limiting diffusion. Writing $S_{i+1} = S_i + \xi_{i+1}$, we obtain

$$H_{i+1}^n - H_i^n = \frac{S_i + \xi_{i+1}}{\sqrt{n}\, t_{i+1}} - \frac{S_i}{\sqrt{n}\, t_i} \quad (175)$$
$$= -\frac{H_i^n - h_0}{n t_i} + \frac{\xi_{i+1}}{\sqrt{n}\, t_i} + o(n^{-1}). \quad (176)$$

NOTE: i just did this computation and it is longer, and needs more to explain. Therefore, conditionally on H_i^n,

$$\mathbb{E}[H_{i+1}^n - H_i^n \mid H_i^n] = \frac{h_0 - H_i^n}{nt_i} + o(n^{-1}), \tag{177}$$

$$\mathrm{Var}(H_{i+1}^n - H_i^n \mid H_i^n) = \frac{1}{nt_i^2} + o(n^{-1}). \tag{178}$$

Thus H^n is an approximation to the diffusion

$$dH_s = \frac{h_0 - H_s}{s}\, ds + \frac{1}{s}\, dB_s,$$

whose time-dependent generator is

$$L_s = \frac{1}{2s^2}\partial_{hh} + \frac{h_0 - h}{s}\partial_h.$$

Let $p_n^H(i, x; j, z)$ denote the transition probabilities of the discrete transformed reference walk $H^{(n)}$. For $\mathbf{i} = (i_1, \ldots, i_k) \in D_k^n$ (recall the definition of D_k^n from (28)) and $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{Z}^k$, define the discrete k-point bridge kernel by

$$\widehat{p}_{k,n}^H(\mathbf{i}, \mathbf{x}; t, h) = \frac{p_n^H(i_0, x_0; i_1, x_1) p_n^H(i_1, x_1; i_2, x_2) \cdots p_n^H(i_k, x_k; i_{k+1}, x_{k+1})}{p_n^H(i_0, x_0; i_{k+1}, x_{k+1})}, \tag{179}$$

where $i_0 = \lfloor nt_0 \rfloor$, and $x_0 = 0$, $\mathrm{i}_{k+1} = \lfloor nt \rfloor$, $\mathrm{x}_{k+1} = \lfloor \sqrt{n}\, t(h - h_0) \rfloor$. Recall from (157) the definition of the kernel

$$K_{h_0}(s, \eta; t, h) = \frac{t}{\sqrt{2\pi(t-s)}} \exp\left\{ -\frac{(th - s\eta - h_0(t-s))^2}{2(t-s)} \right\}. \tag{180}$$

For each i, set

$$t_i = \frac{i(t - t_0)}{n}, \qquad h_{i,x} = h_0 + \frac{x}{\sqrt{n}\, t_i}.$$

The corresponding continuum k-point bridge density is

$$\widehat{\rho}_k^{H;t,h}(\mathbf{r}, \mathbf{y}) = \frac{K_{h_0}(t_0, h_0; r_1, y_1) K_{h_0}(r_1, y_1; r_2, y_2) \cdots K_{h_0}(r_k, y_k; t, h)}{K_{h_0}(t_0, h_0; t, h)}, \tag{181}$$

where

$$\mathbf{r} = (r_1, \ldots, r_k) \in \Delta_k(t_0, t), \qquad \mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k.$$

with $\Delta_k(t_0, t) = \{(r_1, \ldots, r_k) : t_0 < r_1 < \cdots < r_k < t\}$ denotes the time simplex. In particular, when evaluated at the lattice points

$$r_j = t_{i_j}, \qquad y_j = h_{i_j, x_j}, \qquad j = 1, \ldots, k, \tag{182}$$

we obtain

$$\widehat{\rho}_k^{H;t,h} := \widehat{\rho}_k^{H;t,h}(t_{i_1}, h_{i_1,x_1}, \dots, t_{i_k}, h_{i_k,x_k}) = \frac{\prod_{j=0}^{k} K_{h_0}(t_{i_j}, h_{i_j,x_j}; t_{i_{j+1}}, h_{i_{j+1},x_{j+1}})}{K_{h_0}(t_0, h_0; t, h)}, \tag{183}$$

Recall from (73) the definition of the k-fold Wiener integral $W^{\otimes k}(g(\mathbf{r}, \mathbf{y}))$ for a function $g(\mathbf{r}, \mathbf{y}) \in L^2_{[0,1]^k \times \mathbb{R}^k}$, with $\mathbf{r} = (r_1 \cdots r_k)$ and $\mathbf{y} = (y_1 \cdots y_k)$ (recall (182)) . We can now show that the random variable

$$X = \sum_{k=0}^{\infty} W^{\otimes k}(\widehat{\rho}_k^{H;\,t,h}(\mathbf{r}, \mathbf{y})) \tag{184}$$

is a well defined elements of the Fock Space as descibed in (75). Let us temporarily use the notation,

$$r_0 = t_0, \qquad y_0 = h_0, \qquad r_{k+1} = t, \qquad y_{k+1} = h.$$

Since $K_{h_0}(t_0, h_0; t, h) > 0$, the bound we are interested is

$$\left\| \widehat{\rho}_k^{H;\,t,h} \right\|^2_{L^2(\Delta_k(t_0,t) \times \mathbb{R}^k)} = \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} \frac{\prod_{j=0}^{k} K_{h_0}(r_j, y_j; r_{j+1}, y_{j+1})^2}{(K_{h_0}(t_0, h_0; t, h))^2} \, d\mathbf{y} \, d\mathbf{r}. \tag{185}$$

Integration over $\mathbb{R}^k$ produces the bound

$$\int_{\mathbb{R}^k} \frac{\prod_{j=0}^{k} K_{h_0}(r_j, y_j; r_{j+1}, y_{j+1})^2}{(K_{h_0}(t_0, h_0; t, h))^2} \, d\mathbf{y} \;\le\; C^k \prod_{j=0}^{k} (r_{j+1} - r_j)^{-1/2} \prod_{j=1}^{k} r_j \tag{186}$$

where we notice that the factor $r_{k+1} = t$ cancels against the normalizing kernel $K_{h_0}(t_0, h_0; t, h)$ in the denominator, leaving only $\prod_{j=1}^{k} r_j$. Then integrating over the simplex $\Delta_k(t_0, t)$ we achieve the bound

$$\left\| \widehat{\rho}_k^{H;\,t,h} \right\|^2_{L^2(\Delta_k(t_0,t) \times \mathbb{R}^k)} \le C^k \int_{\Delta_k(t_0,t)} \prod_{j=0}^{k} (r_{j+1} - r_j)^{-1/2} \prod_{j=0}^{k} (r_{j+1})^2 \, d\mathbf{r}. \tag{187}$$

Since $t_0 \le r_j \le t$ for each $1 \le j \le k$, we have $\prod_{j=1}^{k} r_j \le t^k$, which is absorbed into the constant C^k (redefining C if necessary). The last integral is a Dirichlet integral:

$$\int_{\Delta_k(t_0,t)} \prod_{j=0}^{k} (r_{j+1} - r_j)^{-1/2} \, d\mathbf{r} = (t - t_0)^{(k-1)/2} \frac{\Gamma(1/2)^{k+1}}{\Gamma((k+1)/2)} \le \frac{C^k}{\Gamma(k/2 + 1)} \tag{188}$$

where the final convergence follows from Stirling's formula and the summability of the last expression in (188). To see the final inequality, recall the Stirling-type asymptotic $\Gamma(x+\frac{1}{2})/\Gamma(x) \sim \sqrt{x}$ as $x \to \infty$; applying this with $x = k/2$ shows $\Gamma(k/2+1)/\Gamma((k+1)/2) = O(\sqrt{k})$, which is in particular $O(C^k)$ for any $C > 1$, giving (188) for a (possibly enlarged) constant $C = C(t_0, t, h_0, h)$, and since $\Gamma(k/2+1)$ grows super-exponentially in k while C^k grows only exponentially, we get

$$\sum_{k\geq 0} \left\| \widehat{\rho}_k^{H;t,h} \right\|^2_{L^2(\Delta_k(t_0,t)\times\mathbb{R}^k)} < \infty, \tag{189}$$

which is precisely the condition needed for X in (184) to be a well-defined element of the Fock space (75); see (E.4) for on the Dirichlet style integrals.

Recall from (180) that for each i , set

$$t_i = \frac{i}{n}, \qquad h_{i,x} = h_0 + \frac{x}{\sqrt{n}\, t_i}.$$

To each lattice point (i, x) we associate the rectangle

$$R_{i,x}^{H,N} = [t_{i-1}, t_i) \times \left[h_0 + \frac{x-1}{\sqrt{N}\, t_i}, h_0 + \frac{x+1}{\sqrt{N}\, t_i}\right). \tag{190}$$

The size of each rectangle is given by

$$|R_{i,x}^{H,N}| = \frac{2}{N^{3/2} t_i}. \tag{191}$$

Note that we obtain the bounds

$$\frac{1}{N^{3/2} t} \leq |R_{i,x}^{H,N}| \leq \frac{1}{N^{3/2} t_0}. \tag{192}$$

and hence the rectangles have size of order comparable to $n^{-3/2}$. Let $\psi \in L^2([t_0, t]\times\mathbb{R})^k)$. For distinct lattice points $(i_1, x_1), \ldots, (i_k, x_k)$ and a rectangle $R_{i_j,x_j}^{H,n}$, define

$$\psi_n\big((i_1, x_1), \ldots, (i_k, x_k)\big) = \frac{1}{|R_{i_j,x_j}^{H,n}|} \int_{R_{i_j,x_j}^{H,n}} \psi(\mathbf{r}, \mathbf{y})\, d\mathbf{r}\, d\mathbf{y}. \tag{193}$$

where

$$\mathbf{r} = (r_1, \ldots, r_k) \in \Delta_k(t_0, t), \qquad \mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k. \tag{194}$$

Recalling that in intermediate disorder we set $\beta_n = \beta n^{-1/4}$; define the k-th discrete U-statistic associated with ψ by

$$S_n^k(\psi) = \beta_n \sum_{i\in D_k^n} \sum_{x\in\mathbb{Z}^k} \psi_n\big((i_1, x_1), \ldots, (i_k, x_k)\big) \prod_{j=1}^{k} \omega(i_j, x_j). \tag{195}$$

This allows us to construct the following U-statistic

$$S^H_{k,n}\left(\widehat{p}^{H;t,h}_k\right) = \sum_{\mathbf{i}\in D^n_k}\sum_{\mathbf{x}\in\mathbb{Z}^k}\widehat{p}^{H;t,h}_{k,n}\big((i_1,x_1),\ldots,(i_k,x_k)\big)\prod_{j=1}^k\omega(i_j,x_j) \tag{196}$$

where we use the notation

$$\widehat{\rho}^{H;t,h}_{k,n}\big((i_1,x_1),\ldots,(i_k,x_k)\big) = \frac{1}{\prod_{j=1}^k|R^{H,n}_{i_j,x_j}|}\int_{\prod_{j=1}^k R^{H,n}_{i_j,x_j}}\widehat{\rho}^{H;t,h}_k(\mathbf{r},\mathbf{y})\,d\mathbf{r}\,d\mathbf{y}. \tag{197}$$

Proposition 5.1 (Fixed-chaos convergence)**.** *Let $k\geq 1$ be fixed. Assume that $\widehat{\rho}^{H;t,h}_k$ is sufficiently regular and square-integrable. Then*

$$n^{-3k/4}\mathcal{S}^H_{k,n}\left(\widehat{\rho}^{H;t,h}_k\prod_{j=1}^k t^{-1}_{i_j}\right)\xrightarrow{(d)}\int_{\Delta_k(t_0,t)}\int_{\mathbb{R}^k}\widehat{\rho}^{H;t,h}_k(\mathbf{r},\mathbf{y})\left(\prod_{j=1}^k r_j^{-1/2}\right)\overline{W}(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{y}), \tag{198}$$

where $\overline{W}(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{y}) = \bigotimes_{j=1}^k\overline{W}(\mathrm{d}r_j,\mathrm{d}y_j)$ is space-time white noise.

Proof. This proof is essential contained in [AKQ14b][Theorem 4.3], however the choice the rectangles $R^{H,n}_{i_j,x_j}$ means that certain steps need to be modified.

Step 1. We first prove the claim in the case $k=1$ and for an indicator function $g(r,y)=\mathbf{1}_A(r,y)$ where

$$A=(a,b]\times(c,d]\subset[t_0,t]\times\mathbb{R}, \tag{199}$$

Recall from (**??**) the definitions of t_i and $h_{i,x}$ above. Then taking $g=g(r,y)$ and k=1, the relevant U-statistic becomes

$$n^{-3/4}\mathcal{S}^H_{1,n}(g\,t_i^{-1}) = n^{-3/4}\sum_{i:\,a<t_i\leq b}\;\sum_{x:\,c<h_{i,x}\leq d}t_i^{-1}\omega(i,x). \tag{200}$$

Since the environment $\omega(i,x)$ is i.i.d, we mean zero and variance one, then the variance of (200) is be given by

$$\mathrm{Var}\left(n^{-3/4}\mathcal{S}^H_{1,n}(g\,t_i^{-1})\right) = n^{-3/2}\sum_{i:\,a<t_i\leq b}\;\sum_{x:\,c<h_{i,x}\leq d}t_i^{-2}. \tag{201}$$

For fixed i, the condition $c<h_{i,x}\leq d$ is equivalent to

$$\sqrt{n}\,t_i(c-h_0)<x\leq\sqrt{n}\,t_i(d-h_0). \tag{202}$$

It follows that

$$\operatorname{Var}\left(n^{-3/4}\mathcal{S}^H_{1,n}(g\,t_i^{-1})\right) = n^{-3/2}\sum_{i:\,a<t_i\le b} t_i^{-2}\left(\sqrt{n}\,t_i(d-c)+O(1)\right) \tag{203}$$

$$= (d-c)n^{-1}\sum_{i:\,a<t_i\le b} t_i^{-1} + o(1). \tag{204}$$

Since $t_i = i/n$, a standard Riemann-sum approximation gives

$$n^{-1}\sum_{i:\,a<t_i\le b} t_i^{-1} \longrightarrow \int_a^b r^{-1}\,dr. \tag{205}$$

Consequently,

$$\operatorname{Var}\left(n^{-3/4}\mathcal{S}^H_{1,n}(g\,t_i^{-1})\right) \longrightarrow (d-c)\int_a^b r^{-1}\,dr. \tag{206}$$

Therefore, by the central limit theorem,

$$n^{-3/4}\mathcal{S}^H_{1,n}(g\,t_i^{-1}) \xrightarrow{(d)} N\left(0, d-c\int_a^b r^{-1}\,dr\right). \tag{207}$$

However calculating the corresponding RHS for (198) with $g(r,y) = \mathbf{1}_A(r,y)$ we obtain that

$$\operatorname{Var}\left(\int_a^b\int_c^d r^{-1/2}\,\overline{W}(dr,dy)\right) = \int_a^b\int_c^d r^{-1}\,dy\,dr \tag{208}$$

$$= (d-c)\int_a^b r^{-1}\,dr. \tag{209}$$

It follows that the limiting Gaussian random variable in (207) has the same mean and covariance as the Wiener integral

$$\int_a^b\int_c^d r^{-1/2}\,\overline{W}(dr,dy).$$

Since both random variables are centred Gaussian, they have the same distribution. Therefore,

$$n^{-3/4}\mathcal{S}^H_{1,n}(g\,t_i^{-1}) \xrightarrow{(d)} \int_a^b\int_c^d r^{-1/2}\,\overline{W}(dr,dy). \tag{210}$$

This proves the $k=1$ convergence for indicator functions.

Step 2. We keep $k=1$. Let $A_1,\ldots,A_m$ be disjoint rectangles of the form considered in Step 1, and let

$$g_\ell = \mathbf{1}_{A_\ell}, \qquad 1\le \ell\le m.$$

Since the environment $\{\omega(i,x)\}_{i,x}$ is i.i.d., and since the rectangles $A_1, \dots, A_m$ are disjoint, the corresponding U-statistics $S^H_{1,n}(g_1 t_i^{-1})$ are asymptotically independent. Hence, by Step 1, we have the joint convergence

$$n^{-3/4}\Big(\mathcal{S}^H_{1,n}(g_1 t_i^{-1}), \dots, \mathcal{S}^H_{1,n}(g_m t_i^{-1})\Big) \xrightarrow{(d)} \Big(\int_{A_1} r^{-1/2}\overline{W}(\mathrm{d}r, \mathrm{d}y), \dots, \int_{A_m} r^{-1/2}\overline{W}(\mathrm{d}r, \mathrm{d}y)\Big). \tag{211}$$

NOTE: Cannot fix overfill

Further for any $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, an application of the Cramér–Wold device gives

$$n^{-3/4}\mathcal{S}^H_{1,n}\left(\left(\sum_{\ell=1}^m \alpha_\ell g_\ell\right) t_i^{-1}\right) \xrightarrow{(d)} \sum_{\ell=1}^m \alpha_\ell \int_{A_\ell} r^{-1/2}\,\overline{W}(dr, dy) \tag{212}$$

$$= \int_{A_\ell} \left(\sum_{\ell=1}^m \alpha_\ell g_\ell(r,y)\right) r^{-1/2}\,\overline{W}(dr, dy). \tag{213}$$

Thus the convergence holds for simple functions supported on finite unions of such rectangles outlined at the begining of the step.

Step 3. We now complete the proof for $k=1$ by a density argument. Choose a sequence of simple functions g_N such that

$$g_N \longrightarrow g \qquad \text{in } L^2([t_0,t]\times\mathbb{R}). \tag{214}$$

By Step 2, for each fixed N,

$$n^{-3/4}\mathcal{S}^H_{1,n}\left(g_N t_i^{-1}\right) \xrightarrow{(d)} \int g_N(r,y) r^{-1/2}\,\overline{W}(dr, dy). \tag{215}$$

Furthermore, by the $L^2([t_0,t]\times\mathbb{R}$ -estimate in (96),we obtain that

$$\mathbb{E}\left[\left| n^{-3/4}\mathcal{S}^H_{1,n}\left((g-g_N) t_i^{-1}\right)\right|^2\right] \longrightarrow 0. \tag{216}$$

Similarly, by the Itô isometry,

$$\mathbb{E}\left[\left|\int (g-g_N)(r,y) r^{-1/2}\,\overline{W}(dr, dy)\right|^2\right] = \|(g-g_N)r^{-1/2}\|^2_{L^2([t_0,t]\times\mathbb{R})} \tag{217}$$

$$\le t_0^{-1}\|g-g_N\|^2_{L^2([t_0,t]\times\mathbb{R})} \longrightarrow 0. \tag{218}$$

Therefore, by the standard approximation theorem for convergence in distribution, see [Jan97, Theorem 4.2]:

$$n^{-3/4}\mathcal{S}^H_{1,n}\left(g t_i^{-1}\right) \xrightarrow{(d)} \int g(r,y) r^{-1/2}\,\overline{W}(dr, dy). \tag{219}$$

This completes the proof for $k = 1$.

Step 4. We now extend the argument to fixed $k > 1$. It is enough to first consider product functions of the form

$$g(\mathbf{r}, \mathbf{y}) = \prod_{j=1}^{k} g_j(r_j, y_j), \tag{220}$$

where each $g_j \in L^2([t_0, t] \times \mathbb{R})$. Finite linear combinations of such product functions are dense in $L^2(([t_0, t] \times \mathbb{R})^k)$.

For functions of the form (220), the k-fold U-statistic factorises, up to the restriction to distinct lattice points, into products of the $k = 1$ sums. Therefore the $k = 1$ convergence from Steps 1–3 gives

$$\begin{aligned} & n^{-3k/4} \mathcal{S}^H_{k,n} \left(g(\mathbf{t}, \mathbf{h}) \prod_{j=1}^{k} t_{i_j}^{-1} \right) \\ & \qquad \xrightarrow{(d)} \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} g(\mathbf{r}, \mathbf{y}) \prod_{j=1}^{k} \left(r_j^{-1/2}\, \overline{W}(dr_j, dy_j) \right). \end{aligned} \tag{221}$$

Finally, by the density of finite linear combinations of product functions, Together with the corresponding discrete L^2-estimate and the Itô isometry, the convergence extends to every $g \in L^2(([t_0, t] \times \mathbb{R})^k)$. □

We now state how the discrete chaos expansion converges to the continuum chaos expansion.

Theorem 5.2. *For any* $\mathbf{g} = (g_0, g_1, \ldots) \in \bigoplus_{k=0}^{\infty} L^2_{\mathrm{sym}}\left([t_0, t]^k \times \mathbb{R}^k\right)$, *we have, as* $n \to \infty$,

$$\begin{aligned} & \sum_{k=0}^{\infty} (t - t_0)^{k/2} n^{-3k/4} \mathcal{S}^H_{k,n} \left(g_k(\mathbf{t}, \mathbf{h}) \prod_{j=1}^{k} t_{i_j}^{-1} \right) \\ & \qquad \xrightarrow{(d)} \sum_{k=0}^{\infty} \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} g_k(\mathbf{r}, \mathbf{y}) \prod_{j=1}^{k} r_j^{-1/2}\, \overline{W}(dr_j, dy_j). \end{aligned} \tag{222}$$

Proof. Recall the isometry from (80), in our present case we have that that the restricted time domain and hence have the following isometry.

$$I^H : \bigoplus_{k=0}^{\infty} L^2_{\mathrm{sym}}([t_0, t]^k \times \mathbb{R}^k) \longrightarrow L^2(\Omega) \tag{223}$$

is an isometry, we temporarily write

$$I^H_k\left(\widehat{\rho}_k^{H;t,h}\right) := \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} \widehat{\rho}_k^{H;t,h}(\mathbf{r}, \mathbf{y}) \prod_{j=1}^{k} r_j^{-1/2}\, \overline{W}(dr_j, dy_j). \tag{224}$$

Recall from (189) that

$$\sum_{k=0}^{\infty}\left\|\widehat{\rho}_k^{H;t,h}\right\|^2_{L^2(\Delta_k(t_0,t)\times\mathbb{R}^k)}<\infty,$$

and hence the tail of the sum converge to zero. Hence, by the isometry (223),

$$\sum_{k=0}^{M} I_k^H\left(\widehat{\rho}_k^{H;t,h}\right)\longrightarrow\sum_{k=0}^{\infty} I_k^H\left(\widehat{\rho}_k^{H;t,h}\right)\tag{225}$$

in $L^2(\Omega)$ as $M\to\infty$. Furthermore, the discrete L^2-estimate (96) can be invoked as for each j we have the bound $t_{i_j}^{-1}\le t_0^{-1}$, this leads to,

$$\sum_{k=0}^{M} n^{-3k/4}\mathcal{S}_{k,n}^H\left(\widehat{\rho}_k^{H;t,h}\prod_{j=1}^{k}t_{i_j}^{-1}\right)\longrightarrow\sum_{k=0}^{\infty} n^{-3k/4}\mathcal{S}_{k,n}^H\left(\widehat{\rho}_k^{H;t,h}\prod_{j=1}^{k}t_{i_j}^{-1}\right)\tag{226}$$

in $L^2(\Omega)$, uniformly in n. By Theorem 5.2, for every fixed M,

$$\sum_{k=0}^{M} n^{-3k/4}\mathcal{S}_{k,n}^H\left(\widehat{\rho}_k^{H;t,h}\prod_{j=1}^{k}t_{i_j}^{-1}\right)\xrightarrow{(d)}\sum_{k=0}^{M} I_k^H\left(\widehat{\rho}_k^{H;t,h}\right).\tag{227}$$

The result now follows from [Jan97, Theorem 4.2]. □

We now consider the modified velocity partition function, which is given by

$$\mathbf{E}_{t,h}^H\left[\prod_{i=1}^{n}\left(1+\beta_n\omega_n(i,S_i)\right)\right]=:\mathfrak{Z}_n^{\mathrm{Vel}}(\omega_n,\beta).\tag{228}$$

where we use the notation $E_{t,h}^H$ so take the expecation of the underlying SRW with repsect to the discrete diffusion as outlined in (174). We note that we are able to write the modified velocity partition function using the following discrete chaos expansion.

$$\begin{aligned}\mathfrak{Z}_n^{\mathrm{Vel}}(\beta)&=\sum_{k=0}^{n}\beta_n^k\sum_{\mathbf{i}\in D_k^n}\sum_{\mathbf{x}\in\mathbb{Z}^k}\prod_{j=1}^{k}\omega(i_j,x_j)\,\widehat{p}_k^{H;t,h}\left(i_j-i_{j-1},\,x_j-x_{j-1}\right)\\&=\sum_{k=0}^{n}\beta^k n^{-k/4}\sum_{\mathbf{i}\in D_k^n}\sum_{\mathbf{x}\in\mathbb{Z}^k}\prod_{j=1}^{k}\omega(i_j,x_j)\,\widehat{p}_k^{H;t,h}\left(i_j-i_{j-1},\,x_j-x_{j-1}\right)\\&=\sum_{k=0}^{n}\beta^k n^{-3k/4}\mathcal{S}_{k,n}^H\left(n^{k/2}\widehat{p}_k^{H;t,h}\right)\end{aligned}\tag{229}$$

From this we have the following theorem.

Theorem 5.3. *Assume that the environment* $(\omega(i,x))_{i\in\mathbb{N},\,x\in\mathbb{Z}}$ *is i.i.d. with mean zero and variance one. Then, under the intermediate disorder scaling*

$$\beta_n = \beta n^{-1/4},$$

the velocity partition function $\mathfrak{Z}_n^{\mathrm{Vel}}(\beta)$ *converges in distribution to* $U(t,h)$*:*

$$\mathfrak{Z}_n^{\mathrm{Vel}}(\beta) \xrightarrow{(d)} U(t,h). \tag{230}$$

Here $U := U(t,h)$ *denotes the mild Itô solution of the transformed stochastic heat equation*

$$\partial_t U = \frac{1}{2t^2}\partial_{hh}U + \frac{h_0 - h}{t}\partial_h U + \beta t^{-1/2} U \overline{W}, \tag{231}$$

with initial condition $U(t_0,\cdot) = \delta_{h_0}$*.*

Remark 1. *We will leave the proof of tightness until Section 9 and prove the convergence of the finite dimensional distributions of* $\mathfrak{Z}_n^{\mathrm{Vel}}(\beta)$ *which converges in distribution to the formal chaos expansion in* (263)*. We delay the proof of the identification of this formal chaos expansion with the solution of the SPDE, this will be given in Section 6.*

Proof. By Theorem 5.1, we have

$$\sum_{k=1}^{\infty} \beta^k n^{-3k/4} \mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} \right) \xrightarrow{(d)} \sum_{k=1}^{\infty} \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} \widehat{\rho}_k^{H;t,h}(\mathbf{r},\mathbf{y}) \prod_{j=1}^{k} \left(r_j^{-1/2}\, \overline{W}(dr_j, dy_j) \right). \tag{232}$$

It therefore suffices to show that the difference between the discrete partition function and the discrete continuum approximation vanishes in $L^2(\Omega,\mathcal{F},Q)$. We estimate

$$\left\| \sum_{k=1}^{\infty} \beta^k n^{-3k/4} \mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} \right) - \sum_{k=1}^{n} \beta^k n^{-3k/4} \mathcal{S}_{k,n}^H \left(n^{k/2} \widehat{p}_{k,n}^{H;t,h} \right) \right\|_{L^2(\Omega,\mathcal{F},Q)} \tag{233}$$

$$\leq \left\| \sum_{k=1}^{n} \beta^k n^{-3k/4} \left[\mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} \right) - \mathcal{S}_{k,n}^H \left(n^{k/2} \widehat{p}_{k,n}^{H;t,h} \right) \right] \right\|_{L^2(\Omega,\mathcal{F},Q)} \tag{234}$$

$$+ \left\| \sum_{k=n+1}^{\infty} \beta^k n^{-3k/4} \mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} \right) \right\|_{L^2(\Omega,\mathcal{F},Q)}. \tag{235}$$

For the term (234), a further application of the triangle inequality shows that

$$\begin{aligned}
&\left\| \sum_{k=1}^{n} \beta^k n^{-3k/4} \left[\mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} \right) - \mathcal{S}_{k,n}^H \left(n^{k/2} \widehat{p}_k^{H;t,h} \right) \right] \right\|_{L^2(\Omega,\mathcal{F},Q)} \\
&\leq \sum_{k=1}^{n} \beta^k n^{-3k/4} \left\| \mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} \right) - \mathcal{S}_{k,n}^H \left(n^{k/2} \widehat{p}_{k,n}^{H;t,h} \right) \right\|_{L^2(\Omega,\mathcal{F},Q)} \\
&\leq \sum_{k=1}^{n} \beta^k \left\| \widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} - n^{k/2} \widehat{p}_{k,n}^H \right\|_{L^2([0,1]^k \times \mathbb{R}^k)} . \qquad (236)
\end{aligned}$$

NOTE: do i need to say something that this is zero for larger n where the last inequality follows from an application of Theorem 4.2. An application of the local limit theorem in Theorem A.4 tells us that

$$\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} - n^{k/2} \widehat{p}_{k,n}^H \longrightarrow 0 \qquad \text{pointwise as } n \to \infty. \qquad (237)$$

However before using the local limit theorem, one must justify the application of the dominated convergence theorem. The bound that we are interested in is given by

$$\sup_{n \in \mathbb{N}} \left\| n^{k/2} \widehat{p}_{k,n}^H \right\|_{L^2([0,1]^k \times \mathbb{R}^k)} . \qquad (238)$$

Using the definition of $n^{k/2}\widehat{p}_{k,n}^H$ we obtain the following set of inequalities

$$\begin{aligned}
&\left\| n^{k/2} \widehat{p}_{k,n}^H (\mathbf{i}, \mathbf{x}; t, h) \right\|_{L^2([0,1]^k \times \mathbb{R}^k)}^2 && (239) \\
&\quad = n^k \sum_{\mathbf{i} \in D_k^n} \sum_{\mathbf{x} \in \mathbb{Z}^k} \left(\widehat{p}_{k,n}^H (\mathbf{i}, \mathbf{x}; t, h) \right)^2 \prod_{j=1}^{k} \frac{2}{n^{3/2} t_{i_j}} && (240) \\
&\quad \leq c^k n^{-k/2} \sum_{\mathbf{i} \in D_k^n} \sum_{\mathbf{x} \in \mathbb{Z}^k} \left(\widehat{p}_{k,n}^H (\mathbf{i}, \mathbf{x}; t, h) \right)^2 && (241) \\
&\quad \leq c^k n^{-k/2} \sum_{\mathbf{i} \in D_k^n} \left(\max_{\mathbf{x} \in \mathbb{Z}^k} \widehat{p}_{k,n}^H (\mathbf{i}, \mathbf{x}; t, h) \right) \sum_{\mathbf{x} \in \mathbb{Z}^k} \widehat{p}_{k,n}^H (\mathbf{i}, \mathbf{x}; t, h) && (242)
\end{aligned}$$

NOTE: should i keep the dependence on x where c^k is a constant depending on t_0 obtained by bounding the factor of $\prod_{j=1}^{k} \frac{2}{t\, i_j}$ by $2/t_0^k$. The first line is justified by noting that, under the diffusive scaling in the definition of (105) that produces a factor of $n^{-3k/2}$ and that the kernel $\widehat{p}_{k,n}^H$ has already been averaged over the rectangles and is therefore constant on each such rectangle. Consequently, the L^2 norm reduces to a sum over the spatial lattice and the index set D_n^k. For each fixed $\mathbf{i}$, we can apply the law of total probability to the final sum in (242), which leads to,

$$
\begin{aligned}
n^{-k/2} & \sum_{\mathbf{i}\in D_k^n} \left(\max_{\mathbf{x}\in\mathbb{Z}^k} \widehat{p}_{k,n}^H(\mathbf{i},\mathbf{x};t,h) \right) \sum_{\mathbf{x}\in\mathbb{Z}^k} \widehat{p}_{k,n}^H(\mathbf{i},\mathbf{x};t,h) \\
&= n^{-k/2} \sum_{\mathbf{i}\in D_k^n} \max_{\mathbf{x}\in\mathbb{Z}^k} \widehat{p}_{k,n}^H(\mathbf{i},\mathbf{x};t,h) \\
&\le c^k n^{-k/2} \sum_{\mathbf{i}\in D_k^n} \prod_{j=1}^k \frac{1}{\sqrt{i_j - i_{j-1}}} \\
&\le c^k \int_{\Delta_k} \prod_{j=1}^k \frac{1}{\sqrt{t_j - t_{j-1}}}\, d\mathbf{t} \\
&\le c^k \left\| \widehat{p}_k^H(\mathbf{i},\mathbf{x};t,h) \right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)} .
\end{aligned} \tag{243}
$$

Where the first inequality follows Remark 3, the second follows from an approximation of the sum to the Riemann integral and the last inequalities follow from the Drichlet type intergral in Subsection E.4. The constant c is still dependent on t_0 but changes line by line in the above set of inequalities. Taking the supremum over in n in (243), we can produce the uniform bound that is needed for the application of the local limit theorem.

$$
\sup_{n\in\mathbb{N}} \left\| n^{k/2} p_{k,h}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x}) \right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)} \le c^k \left\| p_{k,n}^{\mathrm{TILT}}(\mathbf{i},\mathbf{x}) \right\|^2_{L^2([0,1]^k\times\mathbb{R}^k)} . \tag{244}
$$

justifying the application of the dominated convergence theorem in (237). Now we are interested in estimating (235).By another application of the triangle inequality, we obtain

$$
\begin{aligned}
&\sum_{k=n+1}^{\infty} \left\| \beta^k n^{-3k/4} \left[\mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^k r_j^{-1} \right) - \mathcal{S}_{k,n}^H \left(n^{k/2} \widehat{p}_k^{H;t,h} \right) \right] \right\|_{L^2(\Omega,\mathcal{F},Q)} \\
&\le \sum_{k=n+1}^{\infty} \beta^k n^{-3k/4} \left\| \mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^k r_j^{-1} \right) - \mathcal{S}_{k,n}^H \left(n^{k/2} \widehat{p}_k^{H;t,h} \right) \right\|_{L^2(\Omega,\mathcal{F},Q)} .
\end{aligned} \tag{245}
$$

By orthogonality of the homogeneous chaos terms, and repeating the arguments from (121) to (129), we obtain

$$\left\| \sum_{k=n+1}^{\infty} \beta^k n^{-3k/4} \mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} \right) \right\|_{L^2(\Omega,\mathcal{F},Q)}^2$$

$$\leq C \sum_{k=n+1}^{\infty} \beta^{2k} \left\| \widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} r_j^{-1/2} \right\|_{L^2(\Delta_k(t_0,t)\times\mathbb{R}^k)}^2$$

$$\leq C \sum_{k=n+1}^{\infty} (\beta^2 t_0^{-1})^k \left\| \widehat{\rho}_k^{H;t,h} \right\|_{L^2(\Delta_k(t_0,t)\times\mathbb{R}^k)}^2 . \tag{246}$$

Using the Fock-space summability estimate

$$\sum_{k=0}^{\infty} (\beta^2 t_0^{-1})^k \left\| \widehat{\rho}_k^{H;t,h} \right\|_{L^2(\Delta_k(t_0,t)\times\mathbb{R}^k)}^2 < \infty, \tag{247}$$

it follows that the tail of the series vanishes:

$$\sum_{k=n+1}^{\infty} (\beta^2 t_0^{-1})^k \left\| \widehat{\rho}_k^{H;t,h} \right\|_{L^2(\Delta_k(t_0,t)\times\mathbb{R}^k)}^2 \longrightarrow 0 \qquad \text{as } n \to \infty. \tag{248}$$

By (**??**) and the tail estimate above, we have

$$\sum_{k=0}^{\infty} \beta^k n^{-3k/4} \mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} \right)$$

$$\xrightarrow{(d)} \sum_{k=0}^{\infty} \beta^k \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} \widehat{\rho}_k^{H;t,h}(\mathbf{r},\mathbf{y}) \prod_{j=1}^{k} r_j^{-1/2} \, W(dr_j, dy_j). \tag{249}$$

Moreover, by (**??**), as $n \to \infty$,

$$\Bigg\| \sum_{k=1}^{\infty} \beta^k n^{-3k/4} \mathcal{S}_{k,n}^H \left(\widehat{\rho}_k^{H;t,h} \prod_{j=1}^{k} t_{i_j}^{-1} \right) - \sum_{k=1}^{n} \beta^k n^{-3k/4} \mathcal{S}_{k,n}^H \left(n^{k/2} \widehat{p}_k^{H;t,h} \right) \Bigg\|_{L^2(\Omega,\mathcal{F},Q)} \longrightarrow 0. \tag{250}$$

It follows that

$$\mathfrak{Z}_n^{\mathrm{Vel}}(\beta) \xrightarrow{(d)} U(t,h). \tag{251}$$

Theorem 5.4 (Convergence of the velocity partition function)**.** *Assume that there exists $\beta_0 > 0$ such that*

$$\lambda(\beta) := \log Q(e^{\beta\omega}) < \infty \qquad \textit{for all } |\beta| < \beta_0.$$

Set $\beta_n = \beta n^{-1/4}$. Then the renormalised velocity partition function satisfies

$$e^{-n\lambda(\beta_n)} Z_n^{\mathrm{Vel}}(\beta_n) \xrightarrow{(d)} U(t,h), \tag{252}$$

where $U(t,h)$ is given by the chaos expansion

$$U(t,h) = \sum_{k=0}^{\infty} \beta^k \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} \widehat{\rho}_k^{H;t,h}(\mathbf{r},\mathbf{y}) \prod_{j=1}^{k} \left(r_j^{-1/2}\, \overline{W}(dr_j, dy_j) \right). \tag{253}$$

Equivalently, $U(t,h)$ is the mild Itô solution of

$$\partial_t U = \frac{1}{2t^2} \partial_{hh} U + \frac{h_0 - h}{t} \partial_h U + \beta t^{-1/2} U \overline{W}, \tag{254}$$

with initial condition $U(t_0,\cdot) = \delta_{h_0}$.

Proof. Set $\beta_n := \beta n^{-1/4}$. Define a new environment $\widetilde{\omega}_n$ by

$$e^{\beta_n \omega(i,x) - \lambda(\beta_n)} = 1 + \beta_n \widetilde{\omega}_n(i,x). \tag{255}$$

Then

$$\begin{aligned} e^{-n\lambda(\beta_n)} Z_n^{\mathrm{Vel}}(\beta_n) &= \mathbf{E}_{t,h}^{H} \left[\prod_{i=1}^{n} e^{\beta_n \omega(i,S_i) - \lambda(\beta_n)} \right] \\ &= \mathbf{E}_{t,h}^{H} \left[\prod_{i=1}^{n} (1 + \beta_n \widetilde{\omega}_n(i,S_i)) \right] \\ &=: \mathfrak{Z}_n^{\mathrm{Vel}}(\widetilde{\omega}_n, \beta). \end{aligned} \tag{256}$$

We use the notation $\mathfrak{Z}_n^{\mathrm{Vel}}(\widetilde{\omega}_n, \beta)$ to denote the dependence on $\widetilde{\omega}_n$. Expanding the product gives the discrete chaos expansion

$$\mathfrak{Z}_n^{\mathrm{Vel}}(\widetilde{\omega}_n, \beta) = \sum_{k=0}^{n} \beta^k n^{-3k/4} \mathcal{S}_{k,n}^{H} \left(n^{k/2} \widehat{p}_{k,n}^{H;t,h}; \widetilde{\omega}_n \right). \tag{257}$$

Moreover, the reader can directly check from (255) that, $Q[\widetilde{\omega}_n(i,x)] = 0$. Writing

$$\lambda_2(\beta_n) := \lambda(2\beta_n) - 2\lambda(\beta_n),$$

we have

$$\lambda_2(\beta_n) = \beta_n^2 + O(\beta_n^3),$$

and hence by a standard second moment computation

$$Q[\widetilde{\omega}_n(i,x)^2] = 1 + O(\beta_n).$$

After a simple modification of Theorem 4.4, which requires modifying the domain from $[0,t]$ to $[t_0,t]$ **NOTE:** maybe say more, the environment defined in (255) satisfies the required hypothesis.

By the fixed-chaos convergence theorem and the L^2-tail estimates proved above,

$$
\sum_{k=0}^{\infty} \beta^k n^{-3k/4} \mathcal{S}^H_{k,n} \left(\widehat{\rho}^{H;t,h}_k \prod_{j=1}^{k} t_{i_j}^{-1}; \widetilde{\omega}_n \right) \xrightarrow{(d)} \sum_{k=0}^{\infty} \beta^k \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} \widehat{\rho}^{H;t,h}_k(\mathbf{r},\mathbf{y}) \prod_{j=1}^{k} \left(r_j^{-1/2} \, \overline{W}(dr_j, dy_j) \right). \tag{258}
$$

It remains to compare the kernel $n^{k/2}\widehat{p}^{H;t,h}_{k,n}$ with its continuum approximation $\widehat{\rho}^{H;t,h}_k \prod_{j=1}^k t_{i_j}^{-1}$. By the discrete local limit estimate, together with the same L^2-bounds as in Theorem 4.5, mimicking the proof of Theorem 5.1 (lines (236) to (244)) we obtain

$$
\left\| \sum_{k=0}^{n} \beta^k n^{-3k/4} \left[\mathcal{S}^H_{k,n} \left(n^{k/2} \widehat{p}^{H;t,h}_{k,n}; \widetilde{\omega}_n \right) - \mathcal{S}^H_{k,n} \left(\widehat{\rho}^{H;t,h}_k \prod_{j=1}^{k} t_{i_j}^{-1}; \widetilde{\omega}_n \right) \right] \right\|_{L^2(\Omega,\mathcal{F},Q)} \longrightarrow 0. \tag{259}
$$

Furthermore, the corresponding tail estimate (lines gives (245) to (250)) to

$$
\left\| \sum_{k=n+1}^{\infty} \beta^k n^{-3k/4} \mathcal{S}^H_{k,n} \left(\widehat{\rho}^{H;t,h}_k \prod_{j=1}^{k} t_{i_j}^{-1}; \widetilde{\omega}_n \right) \right\|_{L^2(\Omega,\mathcal{F},Q)} \longrightarrow 0. \tag{260}
$$

Combining (257), (258), (259), and (260), we obtain

$$
e^{-n\lambda(\beta_n)} Z^{\mathrm{Vel}}_n(\beta_n) = \mathfrak{Z}^{\mathrm{Vel}}_n(\widetilde{\omega}_n, \beta) \xrightarrow{(d)} \sum_{k=0}^{\infty} \beta^k \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} \widehat{\rho}^{H;t,h}_k(\mathbf{r},\mathbf{y}) \prod_{j=1}^{k} \left(r_j^{-1/2} \, \overline{W}(dr_j, dy_j) \right). \tag{261}
$$

The right-hand side is precisely the chaos expansion of $U(t,h)$. This completes the proof. $\square$

6 Uniqueness of the constructed Stochastic Focker-Plank Equation

In this section we will prove the uniqueness of the solution to the following SPDE which is given by

$$
\partial_t U(t,h) = \frac{1}{2t^2} \partial_{hh} U(t,h) + \frac{h_0 - h}{t} \partial_h U(t,h) + \frac{\beta}{t^{1/2}} U(t,h) \, \overline{W}(t,h). \tag{262}
$$

We saw from (263) that the solution has a well defined chaos expansion which is given by

$$\begin{aligned} U(t,h) = K_{h_0}(t_0,h_0;t,h) &+ \sum_{k=1}^{\infty} \beta^k \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} K_{h_0}(t_0,h_0;s_1,\eta_1) \\ &\times \prod_{r=1}^{k-1} \left[\frac{1}{s_r^{1/2}} K_{h_0}(s_r,\eta_r;s_{r+1},\eta_{r+1}) \right] \frac{1}{s_k^{1/2}} K_{h_0}(s_k,\eta_k;t,h) \\ &\times \overline{W}(ds_1,d\eta_1)\cdots \overline{W}(ds_k,d\eta_k). \end{aligned} \tag{263}$$

where,

$$K_{h_0}(s,\eta;t,h) = \frac{t}{\sqrt{2\pi(t-s)}} \exp\left\{ -\frac{\big(th - s\eta - h_0(t-s)\big)^2}{2(t-s)} \right\}. \tag{264}$$

We would like to claim that, upto a modification on null sets, is the unique solution given by the SPDE (262). To do this we use the methods of [BC95], which produce an appropriately spatially mollified Feynman-Kac formula, which is then proven to be Cauchy in an appropriate probability space. Before we do this, we provide the reader with the a reminder on *cylindrical wiener process.*

Constructing integration against space-time white noise. Let B_t be a cylindrical Wiener process on $L^2(\mathbb{R},dx)$; one may realise B_t as a distribution-valued continuous process by taking

$$\Omega = C([0,\infty),\mathcal{S}'(\mathbb{R})),$$

where $\mathcal{S}'(\mathbb{R})$ is the dual of the Schwartz space $\mathcal{S}(\mathbb{R})$. We equip Ω with the sigma-algebra $\mathcal{F}$ generated by the coordinate maps

$$\omega \mapsto \langle \omega(t), f\rangle, \qquad t \geq 0, \quad f \in \mathcal{S}(\mathbb{R}). \tag{265}$$

where the $\langle\cdot,\cdot\rangle$ pairing on the right-hand side of (266) is the distributional pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$. This allows us to create the probability space $(\Omega,\mathcal{F},W)$. For $\omega\in\Omega$, the value $\omega(t)$ is a tempered distribution. For $f\in\mathcal{S}(\mathbb{R})$, we define $B_t(f)(\omega)$ via (265) by

$$B_t(f)(\omega) = \langle \omega(t), f\rangle, \tag{266}$$

If $(e_k)_{k\geq 1}$ is an orthonormal basis of $L^2(\mathbb{R})$, one may write the expression

$$B_t = \sum_{k=1}^{\infty} \beta_k(t)\, e_k, \tag{267}$$

where $(\beta_k(t))_{k\geq 1}$ are independent Brownian motions associated to the basis $(e_k)_{k\geq 1}$. One can check that (267) is not a well-defined object in $L^2(\mathbb{R})$, however testing (267) against f in a distributional sense produces

$$B_t(f)(\omega) = \sum_{k=1}^{\infty} \beta_k(t,\omega)\langle e_k, f\rangle_{L^2(\mathbb{R})}. \tag{268}$$

The sum in (268) has finite second moment as the orthogonality of the β_k in $L^2(W)$; Parseval's theorem implies that

$$\mathbb{E}_W\left[B_t(f)^2\right] = \sum_{k=1}^{\infty} t\,\langle e_k, f\rangle^2_{L^2(\mathbb{R})} = t\,\|f\|^2_{L^2(\mathbb{R})} < \infty. \tag{269}$$

The probability measure W is the Gaussian measure on Ω uniquely determined by its covariance structure: for all $f, g \in \mathcal{S}(\mathbb{R})$,

$$\mathbb{E}_W[B_t(f)B_s(g)] = (t \wedge s)\langle f, g\rangle_{L^2(\mathbb{R})}. \tag{270}$$

In particular, for each fixed f, $B_t(f)$ is a Brownian motion with variance $t\|f\|^2_{L^2(\mathbb{R})}$. For the rest of this section, we will assume that $f, g \in \mathcal{S}(\mathbb{R})$. Let $(e_i)_{i\geq 1}$ be an orthonormal basis of $L^2(\mathbb{R})$. For $t \geq 0$, define the filtration $\mathcal{F}_t = \sigma\left(B_r(f) : 0 \leq r \leq t,\ f \in L^2(\mathbb{R})\right)$.

Let $(\lambda_t)_{t\geq 0}$ be an $L^2(\mathbb{R})$-valued, $(\mathcal{F}_t)$-adapted process satisfying

$$\mathbb{E}\left[\int_0^t \|\lambda_s\|^2_{L^2(\mathbb{R})}\,ds\right] < \infty.$$

Then the Itô integral of λ with respect to the cylindrical Wiener process B is defined by

$$\int_0^t \langle \lambda_s, dB_s\rangle = \sum_{i=1}^{\infty} \int_0^t \langle \lambda_s, e_i\rangle_{L^2(\mathbb{R})}\,dB_s(e_i), \tag{271}$$

where $B_s(e_i)$, $i \geq 1$, are independent standard Brownian motions.

The Mollified Feynman-Kac Formula for the Focker Plank Equation We first fix $h_0 \in \mathbb{R}$ and let $\beta > 0$ and $t_0 > 0$. We consider the SPDE

$$\partial_t U(t,h) = \frac{1}{2t^2}\partial_{hh}U(t,h) + \frac{h_0 - h}{t}\partial_h U(t,h) + \frac{\beta}{t^{1/2}}U(t,h)\,W(t,h) \tag{272}$$

where $t \in [t_0, T]$. The deterministic part of this operator is given by

$$\mathcal{L}_t = \frac{1}{2t^2}\partial_{hh} + \frac{h_0 - h}{t}\partial_h \tag{273}$$

We can consider the associated diffusion $(H_s)_{s>t_0}$ to the operator in (273), which we claim is given by SDE

$$dH_s = \frac{h_0 - H_s}{s}\,ds + \frac{1}{s}\,dB_s, \qquad H_{t_0} = h, \tag{274}$$

where $(B_s)_{s\geq t_0}$ is a standard Brownian motion and $B_{t_0} = 0$. Suppose that $f \in C^2(\mathbb{R})$ then an application of Ito's lemma provides

$$d(f(H_s)) = f'(H_s)\, dH_s + \frac{1}{2} f''(H_s)\, d\langle H\rangle_s \tag{275}$$

Where $d\langle H\rangle_s$ is the bracket process of the associated diffusion H (see [Gal16], Theorem 4.9). The bracket process $d\langle H\rangle_s$ can be calculated directly from (274) as the first term is of finite variation, leading to $d\langle H\rangle_s = ds/s^2$. Substituting dH_s from (274) into (275), we obtain

$$d(f(H_s)) = \left(f'(H_s)\frac{h_0 - H_s}{s} + \frac{1}{2s^2} f''(H_s)\right) ds + \frac{1}{s} f'(H_s)\, dB_s \tag{276}$$

Comparison of (276) with the operator $\mathcal{L}_t$ in (273) shows that $(H_s)_{s>t_0}$ is the associated diffusion with the operator $\mathcal{L}_t$. Define the sigma algebra $\sigma_t(H) := \sigma\big(H_s : t_0 \leq s \leq t\big)$. Given a function F_s which is predictable with respect to $\sigma_s(H)$, then we are able to integrate F_s with respect to the diffusion H_s. From (274) we have that

$$\int_{t_0}^{t} F_s dH_s = \int_{t_0}^{t} F_s \frac{h_0 - H_s}{s} ds + \int_{t_0}^{t} \frac{F_s}{s} dB_s \tag{277}$$

To compute the covariance structure of the diffusion process H_s, we first remove the time-dependent drift by introducing the rescaled process $Y_s := s(H_s - h_0)$. Using Itô's rule,

$$dY_s = (H_s - h_0)\, ds + s\, dH_s.$$

Substituting in (274), we obtain

$$\begin{aligned} dY_s &= (H_s - h_0)\, ds + s\left(\frac{h_0 - H_s}{s}\, ds + \frac{1}{s}\, dB_s\right) \\ &= (H_s - h_0)\, ds + (h_0 - H_s)\, ds + dB_s \\ &= dB_s. \end{aligned} \tag{278}$$

Therefore writing (278) in intergral form we have

$$Y_s = Y_{t_0} + B_s - B_{t_0} \tag{279}$$

Since $Y_{t_0} = t_0(h - h_0)$, and recalling that $H_{t_0} = h$ we have that

$$Y_s = t_0(h - h_0) + B_s - B_{t_0}. \tag{280}$$

Recalling that $Y_s := s(H_s - h_0)$ this gives the explicit representation

$$H_s = h_0 + \frac{t_0(h - h_0)}{s} + \frac{B_s - B_{t_0}}{s}. \tag{281}$$

We are also interested in the difference process

$$Z_s = H_s^1 - H_s^2 \tag{282}$$

where H_s^1 and H_s^2 are two independent copies of the diffusion H_s. Writing this in differential notation, we obtain

$$dZ_s = -\frac{Z_s}{s}\,ds + \frac{\sqrt{2}}{s}\,d\tilde{B}_s \tag{283}$$

where $\tilde{B}_s$ is another Brownian motion. Here, we have used the property that the difference of two independent Brownian motions, scaled by their diffusion coefficients, results in a new scaled Brownian motion. The bracket process $d\langle z\rangle_s$ can again be calculated from (283) using that the first term is of finite variation, leading to $d\langle Z\rangle_s = 2/s^2\,ds$.

We are interested in the well posedness of the equation given in (262). We will use the outline of [BC95] to guide us through this proof, where will provide the necessary modifications needed. Our proof will rely on a spatial mollification of the Feynman–Kac formula, where H_s plays the role of the path along which the noise is sampled.

We are going to be interested in regularized form of the process B_t which was introduced in (267). Let $h \in C_c^\infty(\mathbb{R})$ such that the $\int_{\mathbb{R}} h(x) = 1$ and that $h(x)$ is even. Now let $\delta > 0$, we consider the mollification $M_\delta(x) = \delta h(\delta x)$. We denote $B_s^\kappa(x)$ denote the spatially mollified noise field, with covariance $\mathbb{E}\left[B_t^\varepsilon(x)B_s^\delta(y)\right] = (t \wedge s)R_{\varepsilon,\delta}(x-y)$, where $R_{\epsilon,\delta} = M_\delta \star M_\epsilon$. Equivalently, its infinitesimal covariance is

$$\mathbb{E}_W\left[dB_s^\varepsilon(x)dB_s^\delta(y)\right] = R_{\epsilon,\delta}(x-y)\,ds. \tag{284}$$

For an α - Holder continuous function φ, define

$$M_\varphi^\kappa(t) := \int_{t_0}^t s^{-1/2}\,dB_s^\kappa(\varphi_s) \tag{285}$$

which is the L^2-limit of the adapted Riemann sums

$$M_\varphi^{\epsilon,n}(t) := \sum_i s_i^{-1/2}\left[B_{s_{i+1}}^\varepsilon(\varphi_{s_i}) - B_{s_i}^\varepsilon(\varphi_{s_i})\right], \tag{286}$$

where $s_i = 2^{-n}$ and $i = 0, \cdots 2^n$ and $t_0 = s_0 < s_1 < \cdots < s_n = t$ In particular, choosing $\varphi_s = H_s$, we obtain the noise integral along the Feynman–Kac diffusion path:

$$M_H^\epsilon(t) := \int_{t_0}^t s^{-1/2}\,dB_s^\varepsilon(H_s). \tag{287}$$

NOTE: I should make a comment to why this is Cauchy **NOTE:** I need to talk about the mesh which again can be considered as the L^2 limit

$$M_H^\kappa(t) = \lim_{|\Pi|\to 0} \sum_i s_i^{-1/2} \left[B^\epsilon_{s_{i+1}}(H_{s_i}) - B^\epsilon_{s_i}(H_{s_i}) \right]. \tag{288}$$

where $|\pi| = \max_{1\le i\le n} |S_i - S_{i-1}|$.

The covariance structure is obtained directly from the covariance of the Hölder-mollified noise. For two α-Hölder continuous functions φ and ψ,

$$\left\langle M_\epsilon^\kappa M_\psi^\delta \right\rangle_t = \int_{t_0}^t \frac{1}{s} R_{\varepsilon,\delta}(\varphi_s - \psi_s)\, ds. \tag{289}$$

Therefore, for two independent copies H^1 and H^2 of the diffusion,

$$\left\langle M_{H^1}^\kappa, M_{H^2}^{\kappa'} \right\rangle_t = \int_{t_0}^t \frac{1}{s} R_{\epsilon,\delta}(H_s^1 - H_s^2)\, ds.$$

On the diagonal, where the same path is paired with itself, we get

$$\langle M_H^\kappa \rangle_t = \int_{t_0}^t \frac{1}{s} R_\epsilon(H_s - H_s)\, ds = \int_{t_0}^t \frac{1}{s} R_\kappa(0)\, ds. = R_\epsilon(0) \log \frac{t}{t_0} \tag{290}$$

NOTE: reread this section Recall the diffusion process H_s as defined in (274); we can consider the associated diffusive bridge. The associated diffusive bridge is the diffusive process defined on the interval $[t_0, t]$ such that $H_{t_0} = z$ and $H_t = h$. At an intermediate time $r \in (t_0, t)$, its one-time marginal density is

$$\mathbb{P}^H_{t_0,z;t,h}(H_r \in dy) = \frac{K_{h_0}(t_0, z; r, y) K_{h_0}(r, y; t, h)}{K_{h_0}(t_0, z; t, h)}\, dy. \tag{291}$$

where we recall the definition of $K_{h_0}(t, z; s, y)$ from (164). The marignal density can be written in the alternative form

$$K_{h_0}(t_0, z; r, y) K_{h_0}(r, y; t, h)\, dy = K_{h_0}(t_0, z; t, h) \mathbb{P}^H_{t_0,z;t,h}(H_r \in dy). \tag{292}$$

Finally, given a finite Radon measure dU_{t_0} on $\mathbb{R}$, we define the finite measure

$$dP^H_{h,t} = \int_{\mathbb{R}} K_{h_0}(t_0, z; t, h)\, \mathbb{P}^H_{t_0,z;t,h}(H_r \in dy)\, dU_{t_0}(z). \tag{293}$$

Here dU_{t_0} denotes the initial measure. We say that the solution $U(t,\cdot)$ attains the initial condition dU_{t_0} if

$$\lim_{t\downarrow t_0} \int_{\mathbb{R}} f(x)\, U(t,x)\, dx = \int_{\mathbb{R}} f(x)\, dU_{t_0}(x), \qquad \text{for every } f \in C_c(\mathbb{R}). \tag{294}$$

Throughout the rest of this section we will use that if H_1 and H_2 are two independent copies of the diffusion given in (274), then $dP^{H_1}_{h,t}$ and $dP^{H_2}_{h,t}$ are two independent copies of the measure in (293). We will use the notation $\mathbb{E}^{H_1}_{h,t}$ and $\mathbb{E}^{H_2}_{h,t}$ to denote expectation with these corresponding measures. Here $E^{H^1}_{t_0,z;\,t,h,}$ denotes expectation with respect to two independent copies of the H-bridge ending at h at time t, with the initial points integrated against the weighted measure $U_0(dz)\,K_{h_0}(t_0,z;t,h)$. Equivalently, for an appropriate functional F,

$$\mathbb{E}^{H^1}_{h,t}E^{H^2}_{h,t}[F(H^1,H^2)] := \int_{\mathbb{R}^2} U_0(dz)U_0(dz')\,K_{h_0}(t_0,z;t,h)K_{h_0}(t_0,z';t,h) \times \mathbb{E}^{H^1}_{t_0,z;\,t,h,}\mathbb{E}^{H^2}_{t_0,z';\,t,h,}\left[F(H^1,H^2)\right]. \tag{295}$$

From this we cosntruct the following mollified Feynman–Kac representation of the SPDE given in .

$$U_\varepsilon(t,h) = \mathbb{E}^{H}_{h,t}\left[U_\varepsilon(t_0,H_t)\exp\left\{\beta M^\epsilon_H(t) - \frac{\beta^2}{2}R_\varepsilon(0)\log\left(\frac{t}{t_0}\right)\right\}\right]. \tag{296}$$

which we will show to be Cauchy in $(\Omega,\mathcal{F},W)$.

Local Times Given any continuous semi-martingale X_t for $t \geq 0$ and $a \in \mathbb{R}$, then there exists an increasing process, $L^a_t(X)$, called the *local time* such that

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sign}(X_s - a)dX_s + L^a_t(X) \tag{297}$$

where $\operatorname{sign}(x) = 1$ if $x > 0$ and $\operatorname{sign}(x) = -1$ if $x \leq 0$. An alternative definition is that one can show that

$$L^a_t(X) = \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_0^t \mathbb{1}_{(a,a+\epsilon)}(X_s)d\langle X,X\rangle_s \tag{298}$$

which can be rigorously proved, see [Gal16]. **NOTE:** Appendix?

Constructing a Cauchy Sequence Through Feynman–Kac Formula We now outline a strategy for proving that the mollified Feynman–Kac given in (296) converge as $\varepsilon \to 0$. We aim to show that $(U_\varepsilon(t,h))_{\varepsilon>0}$ is Cauchy in $L^2(W)$. That is, we want to prove

$$\mathbb{E}_W\left[|U_\varepsilon(t,h) - U_\delta(t,h)|^2\right] \longrightarrow 0 \qquad \text{as } \varepsilon,\delta \to 0. \tag{299}$$

To do this, we will see that it is enough to compute the mixed moments $\mathbb{E}_W\left[U_\varepsilon(t,h)U_\delta(t,h)\right]$. Under the measure of the mollified noise, it follows from covariance structure given in (289) under the mollified noise satisfies

$$\mathbb{E}_W\left[U_\varepsilon(t,h)U_\delta(t,h)\right] = \mathbb{E}^{H^1}_{h,t}E^{H^2}_{h,t}\left[\exp\left\{\beta^2\int_{t_0}^t s^{-1}R_{\varepsilon,\delta}(H^1_s - H^2_s)\,ds\right\}\right], \tag{300}$$

where $R_{\varepsilon,\delta} = \rho_\varepsilon * \rho_\delta$ is the covariance kernel of the two mollified noises. The computation (300) follows from the standard Gaussian exponential identity applied conditionally on the diffusion paths H^1, H^2. After mollification the stochastic integrals are linear functionals of the Gaussian noise and hence are jointly Gaussian. The Wick renormalization subtracts the two self-variance terms, leaving only the covariance which is given by $\beta^2 \int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(H_s^1 - H_s^2)\, ds$. It therefore follows that

$$\begin{aligned}\mathbb{E}_W\left[(U_\varepsilon(t,h) - U_\delta(t,h))^2\right] = \mathbb{E}_{h,t}^{H^1} E_{h,t}^{H^2}\Bigg[&\exp\left\{\beta^2 \int_{t_0}^t s^{-1} R_{\varepsilon,\varepsilon}(H_s^1 - H_s^2)\, ds\right\} \\ &- 2\exp\left\{\beta^2 \int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(H_s^1 - H_s^2)\, ds\right\} \\ &+ \exp\left\{\beta^2 \int_{t_0}^t s^{-1} R_{\delta,\delta}(H_s^1 - H_s^2)\, ds\right\}\Bigg]. \end{aligned} \tag{301}$$

In particular, conditionally on $H_r^1 = \eta$ and $H_r^2 = \eta'$, and on the terminal conditions $H_t^1 = H_t^2 = h$, the process Z is the bridge of this difference diffusion from $\eta - \eta'$ at time r to 0 at time t. Therefore, for any functional F depending only on the difference path,

$$E^Z_{r,\eta-\eta';\,t,0}[F(Z)] = E^{H^1}_{r,\eta;\,t,h} E^{H^2}_{r,\eta';\,t,h}\left[F(H^1 - H^2)\right]. \tag{302}$$

We will also use the corresponding notation $dP^Z_{r,\eta-\eta';\,t,0}$ to denote the associated probability measure with the expectation in (302).

The Cauchy property reduces to understanding the limit of

$$\int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(Z_s)\, ds \qquad \int_{t_0}^t s^{-1} R_{\varepsilon,\varepsilon}(Z_s)\, ds \qquad \int_{t_0}^t s^{-1} R_{\delta,\delta}(Z_s)\, ds \tag{303}$$

as $\varepsilon, \delta \to 0$. Recall the difference process $Z_s = H_s^1 - H_s^2$. given in (283), and the corresponding bracket process calculation $s^{-1}\, ds = \frac{s}{2}\, d\langle Z\rangle_s$. Hence (303) becomes

$$\int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(Z_s)\, ds = \int_{t_0}^t \frac{s}{2} R_{\varepsilon,\delta}(Z_s)\, d\langle Z\rangle_s. \tag{304}$$

Recall the occupational time formula, see [Gal16]. It states that that for $t \geq t_0$, then

$$\int_{t_0}^t \phi(X_s) d\langle X\rangle_s = \int_{\mathbb{R}} \phi(a)(L_t^a(X) - L_{t_0}^a(X)) da \tag{305}$$

holds almost surely for every continuous martingale X_s and non negative function $\phi(x)$ on $\mathbb{R}$. Heuristically we would expect that as $R_{\varepsilon,\delta} \to \delta_0$ as $\varepsilon, \delta \to 0$ and an application of (305) would obtain

$$\int_{t_0}^{t} s^{-1} R_{\varepsilon,\delta}(Z_s)\, ds \longrightarrow \int_{t_0}^{t} \frac{s}{2}\, dL_s^0(Z). \tag{306}$$

However there is a natural issue of defining the measure $dL_s^0(Z)$; further our use of the current form occupational time formula is unjustified, as our function $\phi(s, Z_s)$ has two inputs. We address this in the following lemma.

Lemma 6.1. *Let π be a partition of the interval $[t_0, t]$, then the following modified occuaptional time formula holds.*

$$\int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left(\int_{t_0}^{t} g_\pi(s)\, dL_s^z \right) dz \to \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left(\int_{t_0}^{t} \frac{s}{2}\, dL_s^z \right) dz. \tag{307}$$

as the size of the mesh tends to zero. We can consider integration against dL_s^z in the standard Riemann-Stietljes sense.

Proof. Consider a partition $\pi : \; t_0 = s_0 < s_1 < \cdots < s_n = t$, and define $\pi_{\max} = \max_{1 \le i \le n}(s_i - s_{i-1})$. Applying the standard occupational time formula which is given in (305), we see that

$$\sum_{i=1}^{n} \frac{s_{i-1}}{2} \int_{s_{i-1}}^{s_i} R_{\varepsilon,\delta}(Z_s)\, d\langle Z\rangle_s = \sum_{i=1}^{n} \frac{s_{i-1}}{2} \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \big(L_{s_i}^z - L_{s_{i-1}}^z \big)\, dz. \tag{308}$$

Define the simple function $g_\pi(s) := \sum_{i=1}^{n} \frac{s_{i-1}}{2} \mathbf{1}_{(s_{i-1}, s_i]}(s)$, then we are able to rewrite (308) as

$$\int_{t_0}^{t} g_\pi(s)\, R_{\varepsilon,\delta}(Z_s)\, d\langle Z\rangle_s = \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left(\int_{t_0}^{t} g_\pi(s)\, dL_s^z \right) dz, \tag{309}$$

where the inner integral on the right is a Riemann–Stieltjes integral against the map $s \mapsto L_s^z$, which is justified as this map has finite variation for fixed z, see [Gal16]. As $\pi_{max} \to 0$, $g_\pi(s) \to s/2$ uniformly on $[t_0, t]$. Since $R_{\varepsilon,\delta}$ is bounded and $\langle Z\rangle_s = 2ds/s^2$, which has finite total variation on $[t_0, t]$, an application of the dominated convergence

$$\int_{t_0}^{t} g_\pi(s)\, R_{\varepsilon,\delta}(Z_s)\, d\langle Z\rangle_s \xrightarrow[|\pi| \to 0]{} \int_{t_0}^{t} \frac{s}{2}\, R_{\varepsilon,\delta}(Z_s)\, d\langle Z\rangle_s \qquad \text{a.s.} \tag{310}$$

Again since $s \mapsto L_s^z$ has finite total variation on $[t_0, t]$ and $g_\pi(s) \to s/2$ uniformly,

$$\int_{t_0}^{t} g_\pi(s)\, dL_s^z \xrightarrow[|\pi| \to 0]{} \int_{t_0}^{t} \frac{s}{2}\, dL_s^z. \tag{311}$$

To pass the limit of the RHS of (309) inside the dz-integral, note that for every partition π,

$$\left| \int_{t_0}^{t} g_\pi(s)\, dL_s^z \right| \le \frac{t}{2}\left(L_t^z - L_{t_0}^z\right), \tag{312}$$

and the right-hand side of (312) is integrable against $R_{\varepsilon,\delta}(z)\, dz$ by directly applying the optional time formula (305) directly with $f = R_{\varepsilon,\delta}$ on $[t_0, t]$, which shows $\int_{\mathbb{R}} R_{\varepsilon,\delta}(z)(L_t^z - L_{t_0}^z)\, dz = \int_{t_0}^{t} R_{\varepsilon,\delta}(Z_s)\, d\langle Z\rangle_s < \infty$ a.s. Therefore by considering the by taking $\pi_{max} \to 0$ the LHS and RHS of (309) shows us that

$$\int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left(\int_{t_0}^{t} g_\pi(s)\, dL_s^z \right) dz \xrightarrow[|\pi|\to 0]{} \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left(\int_{t_0}^{t} \frac{s}{2}\, dL_s^z \right) dz. \tag{313}$$

as claimed. $\square$

We now use (6.1) to prove the following.

Theorem 6.2. *Let $p \in (1, \infty)$ and denote $\mu_{r,t} := d\mathbb{P}^Z_{r,\eta-\eta';\,t,0}$. We have the following convergence in $L^p(\mu_{r,t})$.*

$$\int_{t_0}^{t} s^{-1} R_{\varepsilon,\delta}(Z_s)\, ds \longrightarrow \int_{t_0}^{t} \frac{s}{2}\, dL_s^0(Z) \qquad \text{as } \varepsilon, \delta \to 0. \tag{314}$$

Proof. Recall from (304) that

$$\int_{t_0}^{t} \frac{s}{2} R_{\varepsilon,\delta}(Z_s)\, d\langle Z\rangle_s = \int_{t_0}^{t} s^{-1} R_{\varepsilon,\delta}(Z_s)\, ds \tag{315}$$

Therefore by a direct application of (6.1), with π being the partition of $[t_0, t]$, we have that as the mesh size tends to zero,

$$\int_{t_0}^{t} \frac{s}{2} R_{\varepsilon,\delta}(Z_s)\, d\langle Z\rangle_s \to \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left(\int_{t_0}^{t} \frac{s}{2}\, dL_s^z(Z) \right) dz. \tag{316}$$

Define

$$G_t(z) := \int_{t_0}^{t} \frac{s}{2}\, dL_s^z(Z).$$

Since $s \mapsto s/2$ is continuously differentiable, an application of Ito's Lemma gives us

$$G_t(z) = \frac{t}{2} L_t^z(Z) - \frac{1}{2} \int_{t_0}^{t} L_s^z(Z)\, ds. \tag{318}$$

Where we have used the fact that the local time is normalized to start from t_0, so that $L^z_{t_0}(Z) = 0$. We now estimate the quantity of interest, by first applying Minkowski's integral inequality,

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} R_{\varepsilon,\delta}(z)\big(G_t(z) - G_t(0)\big)\, dz \right\|_{L^p(d\mathbb{P}^Z_{r,\eta-\eta';t,0})} \\
&\leq \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left\| G_t(z) - G_t(0) \right\|_{L^p(d\mathbb{P}^Z_{r,\eta-\eta';t,0})} dz. \qquad (319)
\end{aligned}$$

Recalling the definition of Z_s from (281), which states that

$$dZ_s = -\frac{Z_s}{s}\, ds + \frac{\sqrt{2}}{s}\, d\tilde{B}_s \qquad (320)$$

we have that Z is the difference of two independent copies of the diffusion $(H_s)_{s\geq t_0}$. Using the explicit representation in (281), since $1/s$ is bounded and smooth on the interval $[t_0, t]$, we note that

$$sZ_s = t_0 Z_{t_0} + \sqrt{2}(B_s - B_{t_0}). \qquad (321)$$

Thus Z is obtained from Brownian motion by a deterministic space-time transformation. Since $s \in [t_0, t]$ is bounded above and bounded away from zero, this transformation does not change the relevant local-time integrability estimates. In particular, the local time of Z satisfies the same type of spatial modulus and exponential moment estimates as Brownian local time. Therefore, for sufficiently small $c > 0$, by the exponential local-time estimate used in Lemma 3.2 of [BC95] holds. Therefore for every $p \geq 1$ and every $\alpha < 1/2$, there exists a constant $C_{p,\alpha,t,t_0} < \infty$ such that For every $p \geq 1$ and every $\alpha \in (0, 1/2)$, we have

$$\left\| G_t(z) - G_t(0) \right\|_{L^p(d\mathbb{P}^Z_{r,\eta-\eta';t,0})} \leq C_{p,\alpha,t,t_0} |z|^{\alpha}. \qquad (322)$$

Consequently, the modified local time $G_t(z)$ in (318) satisfies the same spatial Hölder estimate. Hence, returning to (319), we obtain

$$\int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left\| G_t(z) - G_t(0) \right\|_{L^p(d\mathbb{P}^Z_{r,\eta-\eta';t,0})} dz \leq \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) C_{p,\alpha,t,t_0} |z|^{\alpha}\, dz. \quad (323)$$

Recall that

$$R_{\varepsilon,\delta} = P_\varepsilon * \widetilde{P}_\delta,$$

where

$$P_\varepsilon(x) = \frac{1}{\varepsilon} P\left(\frac{x}{\varepsilon}\right), \qquad \widetilde{P}_\delta(x) = P_\delta(-x).$$

Since P is compactly supported on $[-1, 1]$, we have

$$\operatorname{supp}(P_\varepsilon) \subseteq [-\varepsilon, \varepsilon], \qquad \operatorname{supp}(\widetilde{P}_\delta) \subseteq [-\delta, \delta].$$

Therefore

$$\operatorname{supp}(R_{\varepsilon,\delta}) = \operatorname{supp}(P_\varepsilon * \widetilde{P}_\delta) \subseteq [-(\varepsilon+\delta), \varepsilon+\delta].$$

Returning to (322), we obtain

$$\int_{\mathbb{R}} R_{\varepsilon,\delta}(z) C_{p,\alpha,t,t_0} |z|^\alpha \, dz \le C_{p,\alpha,t,t_0} (\varepsilon+\delta)^\alpha \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \, dz. \tag{324}$$

Since $R_{\varepsilon,\delta}$ has total mass one, it follows that

$$\int_{\mathbb{R}} R_{\varepsilon,\delta}(z) C_{p,\alpha,t,t_0} |z|^\alpha \, dz \le C_{p,\alpha,t,t_0} (\varepsilon+\delta)^\alpha \longrightarrow 0 \tag{325}$$

as $\varepsilon, \delta \to 0$. □

We now provide the appropriate estimates needed for uniform intergrability.

Lemma 6.3. *There exist constants $c > 0$ such that*

$$\sup_{h\in\mathbb{R}} \mathbb{E}^{H^1}_{h,t} E^{H^2}_{h,t} \left[\exp\left\{ c \int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(Z_s) \, ds \right\} \right] < \infty. \tag{326}$$

Proof. We recall the occupation time formula, derived in (316). Using this, it follows that

$$\begin{aligned} &\mathbb{E}^{H^1}_{h,t} E^{H^2}_{h,t} \left[\exp\left\{ c \int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(Z_s) \, ds \right\} \right] \\ &\qquad = \mathbb{E}^{H^1}_{h,t} E^{H^2}_{h,t} \left[\exp\left\{ c \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left(\int_{t_0}^t \frac{s}{2} \, dL^z_s(Z) \right) dz \right\} \right]. \end{aligned} \tag{327}$$

Using that $s \le t$ for $s \in [t_0, t]$, and that $dL^z_s(Z)$ is a nonnegative measure in s, we obtain

$$\begin{aligned} \int_{t_0}^t \frac{s}{2} \, dL^z_s(Z) &\le \frac{t}{2} \int_{t_0}^t dL^z_s(Z) \\ &= \frac{t}{2} \left(L^z_t(Z) - L^z_{t_0}(Z) \right) \\ &= \frac{t}{2} L^z_t(Z). \end{aligned} \tag{328}$$

where again we are using the convention that $L^z_{t_0}(Z) = 0$. Using (328) in (327), we obtain

$$\begin{aligned} &\mathbb{E}^{H^1}_{h,t} E^{H^2}_{h,t} \left[\exp\left\{ c \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \left(\int_{t_0}^t \frac{s}{2} \, dL^z_s(Z) \right) dz \right\} \right] \\ &\qquad \le \mathbb{E}^{H^1}_{h,t} E^{H^2}_{h,t} \left[\exp\left\{ \frac{ct}{2} \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) L^z_t(Z) \, dz \right\} \right]. \end{aligned} \tag{329}$$

Since $R_{\varepsilon,\delta} \geq 0$ and $\int_{\mathbb{R}} R_{\varepsilon,\delta}(z)\,dz = 1$, Jensen's inequality gives

$$\begin{aligned}
&\mathbb{E}_{h,t}^{H^1} E_{h,t}^{H^2}\left[\exp\left\{\frac{ct}{2}\int_{\mathbb{R}} R_{\varepsilon,\delta}(z) L_t^z(Z)\,dz\right\}\right] \\
&\quad \leq \int_{\mathbb{R}} R_{\varepsilon,\delta}(z) \mathbb{E}_{h,t}^{H^1} E_{h,t}^{H^2}\left[\exp\left\{\frac{ct}{2} L_t^z(Z)\right\}\right] dz \\
&\quad \leq \sup_{h\in\mathbb{R}} \mathbb{E}_{h,t}^{H^1} E_{h,t}^{H^2}\left[\exp\left\{\frac{ct}{2} L_t^z(Z)\right\}\right].
\end{aligned} \tag{330}$$

It remains to show that, for sufficiently small $c > 0$,

$$\sup_{z\in\mathbb{R}} \mathbb{E}_{h,t}^{H^1} E_{h,t}^{H^2}\left[\exp\left\{\frac{ct}{2} L_t^z(Z)\right\}\right] < \infty. \tag{331}$$

Again the diffusion process Z is obtained from Brownian motion by a deterministic space-time transformation. In particular, its local time satisfies the same type of exponential moment estimates as Brownian local time as s is bounded in the interval $[t_0, t]$. Therefore, for sufficiently small $c > 0$, (331) hold, from the exponential local-time estimate used in Lemma 3.2 of [BC95].

We are now able to prove that the sequence $U_{\varepsilon,\delta}(t,h)$ as constructed in the mollified Feynman-Kac formula is Cauchy in $L^2(W)$, we have that

$$\begin{aligned}
&\mathbb{E}_W\left[\left(U_\varepsilon(t,h) - U_\delta(t,h)\right)^2\right] \\
&\quad = \mathbb{E}_{h,t}^{H^1} E_{h,t}^{H^2}\left[\exp\left\{\beta^2\int_{t_0}^t s^{-1} R_{\varepsilon,\varepsilon}(Z_s)\,ds\right\} + \exp\left\{\beta^2\int_{t_0}^t s^{-1} R_{\delta,\delta}(Z_s)\,ds\right\}\right. \\
&\qquad\qquad \left. -2\exp\left\{\beta^2\int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(Z_s)\,ds\right\}\right] \\
&\quad = \mathbb{E}_{h,t}^{H^1} E_{h,t}^{H^2}\left[\left(\exp\left\{\beta^2\int_{t_0}^t s^{-1} R_{\varepsilon,\varepsilon}(Z_s)\,ds\right\} - \exp\left\{\beta^2\int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(Z_s)\,ds\right\}\right)\right. \\
&\qquad\qquad \left. + \left(\exp\left\{\beta^2\int_{t_0}^t s^{-1} R_{\delta,\delta}(Z_s)\,ds\right\} - \exp\left\{\beta^2\int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(Z_s)\,ds\right\}\right)\right],
\end{aligned} \tag{332}$$

Looking at the expression (332), we can see that a necessary condition to prove that the sequence $U_\varepsilon(t,h)$ is to prove that

$$\mathbb{E}_{h,t}^{H^1} E_{h,t}^{H^2}\left[\left(\exp\left\{\beta^2\int_{t_0}^t s^{-1} R_{\varepsilon,\varepsilon}(Z_s)\,ds\right\} - \exp\left\{\beta^2\int_{t_0}^t s^{-1} R_{\varepsilon,\delta}(Z_s)\,ds\right\}\right)\right. \tag{333}$$

goes to zero as $\delta \to 0$ and $\varepsilon \to 0$. Consequently we look at

$$
\begin{aligned}
&\mathbb{E}_{h,t}^{H^1} E_{h,t}^{H^2}\left[\left|\exp\left\{\beta^2\int_{t_0}^t s^{-1}R_{\varepsilon,\varepsilon}(Z_s)\,ds\right\} - \exp\left\{\beta^2\int_{t_0}^t s^{-1}R_{\varepsilon,\delta}(Z_s)\,ds\right\}\right|\right] \\
&\quad\le \left\|\exp\left\{\beta^2\int_{t_0}^t s^{-1}R_{\varepsilon}(Z_s)\,ds\right\} + \exp\left\{\beta^2\int_{t_0}^t s^{-1}R_{\varepsilon,\delta}(Z_s)\,ds\right\}\right\|_{L^2(\mathbb{P}_{H^1,H^2})} \qquad (334)\\
&\qquad\quad \times \left\|\beta^2\int_{t_0}^t s^{-1}\left(R_{\varepsilon}(Z_s) - R_{\varepsilon,\delta}(Z_s)\right)\,ds\right\|_{L^2(\mathbb{P}_{H^1,H^2})}. \qquad (335)
\end{aligned}
$$

By Theorem 6.3, the first factor (334) is bounded uniformly in $\varepsilon, \delta > 0$. Furthermore, Theorem 6.2 implies that

$$
\left\|\beta^2\int_{t_0}^t s^{-1}\big(R_{\varepsilon}(Z_s) - R_{\varepsilon,\delta}(Z_s)\big)\,ds\right\|_{L^2(\mathbb{P}_{H^1,H^2})} \longrightarrow 0
$$

as $\delta \to 0$, followed by $\varepsilon \to 0$. Consequently, the right-hand side of (334)–(335) converges to zero, proving the desired convergence.

We now are in position to prove that the limit of the Cauchy sequence constructed is the unique solution of (343); before we state our Theorem, we need one Lemma.

Lemma 6.4. *Let $T > t_0 > 0$. There exists a constant $C = C(t_0, T) > 0$ such that, for every $t_0 \le r < t \le T$ and $h \in \mathbb{R}$,*

$$
\int_{\mathbb{R}} K_{h_0}(r, y; t, h)^2\,dy \le C\,(t-r)^{-1/2}. \tag{336}
$$

Proof. Recall that

$$
K_{h_0}(r, y; t, h) = \frac{t}{\sqrt{2\pi(t-r)}}\exp\left\{-\frac{(th - ry - h_0(t-r))^2}{2(t-r)}\right\}. \tag{337}
$$

Consequently,

$$
\int_{\mathbb{R}} K_{h_0}(r, y; t, h)^2\,dy = \frac{t^2}{2\pi(t-r)}\int_{\mathbb{R}}\exp\left\{-\frac{(th - ry - h_0(t-r))^2}{t-r}\right\}\,dy. \tag{338}
$$

Making the change of variables

$$
u = th - ry - h_0(t-r), \qquad du = -r\,dy,
$$

gives

$$\int_{\mathbb{R}} K_{h_0}(r, y; t, h)^2\, dy = \frac{t^2}{2\pi r(t-r)} \int_{\mathbb{R}} \exp\left\{-\frac{u^2}{t-r}\right\} du \tag{339}$$

$$= \frac{t^2}{2\pi r(t-r)} \sqrt{\pi(t-r)} \tag{340}$$

$$= \frac{t^2}{2\sqrt{\pi}\, r}(t-r)^{-1/2}. \tag{341}$$

Since $r \geq t_0$ and $t \leq T$, it follows that

$$\int_{\mathbb{R}} K_{h_0}(r, y; t, h)^2\, dy \leq \frac{T^2}{2\sqrt{\pi}\, t_0}(t-r)^{-1/2}. \tag{342}$$

Theorem 6.5. *Let $t > t_0 > 0$, and consider the formal SPDE*

$$\partial_t U(t, h) = \frac{1}{2t^2} \partial_{hh} U(t, h) + \frac{h_0 - h}{t} \partial_h U(t, h) + \frac{\beta}{t^{1/2}} U(t, h)\, \overline{W}(t, h), \tag{343}$$

with initial condition $U(t_0, \cdot) = \delta_{h_0}$. Let $K_{h_0}(s, \eta; t, h)$ denote the transition density of the diffusion with time-dependent generator

$$\mathcal{L}_s = \frac{1}{2s^2} \partial_{hh} + \frac{h_0 - h}{s} \partial_h.$$

Assume limiting field $U(t, h)$ satisfies integrability condition: for every $t > t_0$,

$$\sup_{t \in [t_0, T]} \sup_{h \in \mathbb{R}} \int_{t_0}^{t} \int_{t_0}^{r} \int_{\mathbb{R}^2} K_{h_0}(r, y; t, h)^2 r^{-1} K_{h_0}(q, \xi; r, y)^2 q^{-1} \times \mathbb{E}\left[|U(q, \xi)|^2\right] d\xi\, dy\, dq\, dr < \infty. \tag{344}$$

Then $U(t, h)$ admits the Wiener chaos expansion

$$\begin{aligned} U(t, h) = K_{h_0}(t_0, h_0; t, h) & \\ + \sum_{k=1}^{\infty} \beta^k \int_{\Delta_k(t_0, t)} \int_{\mathbb{R}^k} & K_{h_0}(t_0, h_0; s_1, \eta_1) \\ & \times \prod_{r=1}^{k-1} \left[s_r^{-1/2} K_{h_0}(s_r, \eta_r; s_{r+1}, \eta_{r+1}) \right] \\ & \times s_k^{-1/2} K_{h_0}(s_k, \eta_k; t, h)\, \overline{W}(ds_1, d\eta_1) \cdots \overline{W}(ds_k, d\eta_k), \end{aligned} \tag{345}$$

where

$$\Delta_k(t_0, t) = \{(s_1, \ldots, s_k) : t_0 < s_1 < \cdots < s_k < t\}.$$

The series in (345) converges in $L^2(W)$. Moreover, $U(t, h)$ is the unique L^2-mild solution of (343) in the class of adapted random-field solutions satisfying the corresponding second-moment bounds in (344).

Proof. We break this into a four-step process for the convenience of the reader.

Step 1: We first work with the mollified equation rather than the SPDE directly, since the mollified equation is classically well defined. We then represent the mollified solution at an intermediate time using the Feynman-Kac formula and show it satisfies the mollified mild solution.

Step 2: We then remove the mollification by taking $\epsilon \to 0$ in $L^2(W)$

Step 3: We then show uniquess of the mild solution.

Step 4: We then explain why we can consider the chaos expansion in (345) as a solution of the SPDE in (343).

Throughout, the diffusion bridge H with generator $\mathcal{L}_s$ is taken independent of the driving noise $\overline{W}$ (and its mollifications $\overline{W}_\varepsilon$). Recall the defintion of the mollification $M_\varepsilon(x)$, which has compact support, total mass one and is even. We define the spatially mollified space-time white noise by

$$\overline{W}_\varepsilon(ds,h) = \int_{\mathbb{R}} M_\varepsilon(h-y)\,\overline{W}(ds,dy). \tag{346}$$

Then $\overline{W}_\varepsilon$ is smooth in h and has covariance

$$\mathbb{E}\big[\overline{W}_\varepsilon(ds,h)\overline{W}_\delta(dt,h')\big] = \delta(s-t)\,R_{\varepsilon,\delta}(h-h')\,ds\,dt, \tag{347}$$

Step 1 Consider the spatially mollified and Wick-renormalised equation

$$\begin{aligned}\partial_t U_\varepsilon(t,h) = \frac{1}{2t^2}\partial_{hh}U_\varepsilon(t,h) + \frac{h_0-h}{t}\partial_h U_\varepsilon(t,h) + \frac{\beta}{t^{1/2}}U_\varepsilon(t,h)\overline{W}_\varepsilon(t,h)\\ - \frac{\beta^2}{2t}R_\varepsilon(0)U_\varepsilon(t,h).\end{aligned} \tag{348}$$

Its mild formulation is

$$\begin{aligned}U_\varepsilon(t,h) = &\int_{\mathbb{R}} U_0(dz)K_{h_0}(t_0,z;t,h)\\ &+ \beta\int_{t_0}^t\int_{\mathbb{R}} K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1/2}\overline{W}_\varepsilon(r,y)\,dy\,dr\\ &- \frac{\beta^2}{2}R_\varepsilon(0)\int_{t_0}^t\int_{\mathbb{R}} K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1}\,dy\,dr.\end{aligned} \tag{349}$$

NOTE: explain why this is the mild solution

Using the Feynman–Kac representation with initial measure $U_0(dz)$, at the intermediate point (r,y), we have

$$U_\varepsilon(r,y) = \int_{\mathbb{R}} U_0(dz)\,K_{h_0}(t_0,z;r,y)\mathbb{E}^H_{t_0,z;r,y}\left[\mathcal{E}_\varepsilon(r)\right], \tag{350}$$

where we use the temporary notation

$$\mathcal{E}_\varepsilon(r) := \exp\left\{\beta M_H^\epsilon(t) - \frac{\beta^2}{2} R_\varepsilon(0)\log\left(\frac{r}{t_0}\right)\right\}. \tag{351}$$

Substituting (350) into the stochastic term in (349) gives

$$\begin{aligned}\beta \int_{t_0}^{t}\int_{\mathbb{R}} & K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1/2}\overline{W}_\varepsilon(r,y)\,dy\,dr \\ &= \beta\int_{\mathbb{R}} U_0(dz)\int_{t_0}^{t}\int_{\mathbb{R}} K_{h_0}(t_0,z;r,y)K_{h_0}(r,y;t,h) \\ &\qquad \times \mathbb{E}^H_{t_0,z;r,y}\left[\mathcal{E}_\varepsilon(r)\right] r^{-1/2}\overline{W}_\varepsilon(r,y)\,dy\,dr. \end{aligned} \tag{352}$$

For $r \in (t_0,t)$, the full H-bridge from (t_0,z) to (t,h) has one-time marginal density

$$\mathbb{P}^H_{t_0,z;t,h}(H_r \in dy) = \frac{K_{h_0}(t_0,z;r,y)K_{h_0}(r,y;t,h)}{K_{h_0}(t_0,z;t,h)}\,dy. \tag{353}$$

Which can be alternatively written as

$$K_{h_0}(t_0,z;r,y)K_{h_0}(r,y;t,h)\,dy = K_{h_0}(t_0,z;t,h)\mathbb{P}^H_{t_0,z;t,h}(H_r \in dy). \tag{354}$$

Moreover, if $F_r(H)$ depends only on the path up to time r, then the Markov property of the bridge gives

$$\mathbb{E}^H_{t_0,z;t,h}\left[F_r(H) \mid H_r = y\right] = \mathbb{E}^H_{t_0,z;r,y}\left[F_r(H)\right]. \tag{355}$$

Since $\mathcal{E}_\varepsilon(r)$ depends only on the path up to time r, we obtain

$$\mathbb{E}^H_{t_0,z;t,h}\left[\mathcal{E}_\varepsilon(r) \mid H_r = y\right] = \mathbb{E}^H_{t_0,z;r,y}\left[\mathcal{E}_\varepsilon(r)\right]. \tag{356}$$

Combining (354) and (356), and using that the mollified noise can be evaluated along the continuous bridge path H, we get

$$\begin{aligned}\int_{\mathbb{R}} K_{h_0}(t_0,z;r,y)&K_{h_0}(r,y;t,h)\mathbb{E}^H_{t_0,z;r,y}\left[\mathcal{E}_\varepsilon(r)\right]\overline{W}_\varepsilon(r,y)\,dy \\ &= K_{h_0}(t_0,z;t,h)\mathbb{E}^H_{t_0,z;t,h}\left[\mathcal{E}_\varepsilon(r)\overline{W}_\varepsilon(r,H_r)\right]. \end{aligned} \tag{357}$$

Hence returning to 352 we obtain that

$$\begin{aligned}\beta\int_{t_0}^{t}\int_{\mathbb{R}} & K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1/2}\overline{W}_\varepsilon(r,y)\,dy\,dr \\ &= \beta\int_{\mathbb{R}} U_0(dz)K_{h_0}(t_0,z;t,h)\mathbb{E}^H_{t_0,z;t,h}\left[\int_{t_0}^{t} r^{-1/2}\mathcal{E}_\varepsilon(r)\overline{W}_\varepsilon(r,H_r)\,dr\right]. \end{aligned} \tag{358}$$

The same process can be done to the Feynman–Kac representation into for the renormalization term given in (349) this gives

$$\begin{aligned}&\frac{\beta^2}{2}R_\varepsilon(0)\int_{t_0}^t\int_{\mathbb{R}}K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1}\,dy\,dr\\&\quad=\frac{\beta^2}{2}R_\varepsilon(0)\int_{\mathbb{R}}U_0(dz)K_{h_0}(t_0,z;t,h)\mathbb{E}^H_{t_0,z;t,h}\left[\int_{t_0}^t r^{-1}\mathcal{E}_\varepsilon(r)\,dr\right].\end{aligned} \tag{359}$$

Now observe that, pathwise along the bridge,

$$\frac{d}{dr}\mathcal{E}_\varepsilon(r)=\beta r^{-1/2}\mathcal{E}_\varepsilon(r)\overline{W}_\varepsilon(r,H_r)-\frac{\beta^2}{2}R_\varepsilon(0)r^{-1}\mathcal{E}_\varepsilon(r). \tag{360}$$

Therefore integrating over $r\in[t_0,t]$ gives us

$$\begin{aligned}\mathcal{E}_\varepsilon(t)-1=&\int_{t_0}^t\beta r^{-1/2}\mathcal{E}_\varepsilon(r)\overline{W}_\varepsilon(r,H_r)\,dr\\&-\frac{\beta^2}{2}R_\varepsilon(0)\int_{t_0}^t r^{-1}\mathcal{E}_\varepsilon(r)\,dr.\end{aligned} \tag{361}$$

Combining (358), (359), and (361), we obtain that the mollified mild solution in (349) can be written as

$$\begin{aligned}&\beta\int_{t_0}^t\int_{\mathbb{R}}K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1/2}\overline{W}_\varepsilon(r,y)\,dy\,dr\\&\quad-\frac{\beta^2}{2}R_\varepsilon(0)\int_{t_0}^t\int_{\mathbb{R}}K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1}\,dy\,dr\\&=\int_{\mathbb{R}}U_0(dz)K_{h_0}(t_0,z;t,h)\mathbb{E}^H_{t_0,z;t,h}\left[\mathcal{E}_\varepsilon(t)-1\right].\end{aligned} \tag{362}$$

Hence

$$\begin{aligned}&\beta\int_{t_0}^t\int_{\mathbb{R}}K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1/2}\overline{W}_\varepsilon(r,y)\,dy\,dr\\&\quad-\frac{\beta^2}{2}R_\varepsilon(0)\int_{t_0}^t\int_{\mathbb{R}}K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1}\,dy\,dr\\&=U_\varepsilon(t,h)-\int_{\mathbb{R}}U_0(dz)K_{h_0}(t_0,z;t,h).\end{aligned} \tag{363}$$

This is precisely the mild formulation (349).

Step 2 It remains to justify the convergence of the stochastic convolutions. Set

$$\Phi_\varepsilon(r,y)=K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1/2},\qquad \Phi(r,y)=K_{h_0}(r,y;t,h)U(r,y)r^{-1/2}.$$

Then using (346)

$$\int_{t_0}^{t}\int_{\mathbb{R}} \Phi_\varepsilon(r,y)\overline{W}_\varepsilon(r,y)\,dy\,dr$$

$$= \int_{t_0}^{t}\int_{\mathbb{R}} \left(\Phi_\varepsilon(r,\cdot) * M_\varepsilon\right)(z)\,\overline{W}(dr,dz), \tag{364}$$

Hence, by the Itô isometry,

$$\mathbb{E}\left[\left|\int_{t_0}^{t}\int_{\mathbb{R}} \left[\big(\Phi_\varepsilon(r,\cdot) * M_\varepsilon\big)(z) - \Phi(r,z)\right]\,\overline{W}(dr,dz)\right|^2\right]$$

$$= \mathbb{E}\int_{t_0}^{t} \|\Phi_\varepsilon(r,\cdot) * M_\varepsilon - \Phi(r,\cdot)\|^2_{L^2(\mathbb{R})}\,dr. \tag{365}$$

We split

$$\Phi_\varepsilon * M_\varepsilon - \Phi = (\Phi_\varepsilon - \Phi) * M_\varepsilon + (\Phi * M_\varepsilon - \Phi). \tag{366}$$

Using (366) in (365) that the first term tends to zero because convolution is a contraction on $L^2(\mathbb{R})$ and that in this section we have already proven that $U_\varepsilon \to U$ in L^2 with the kernel weight $K^2_{h_0} r^{-1}$. The second term tends to zero because M_ε is an approximation of the identity in L^2 For the second term in (366): by (**??**) and Lemma 6.4, $\Phi(r,\cdot) \in L^2(\mathbb{R})$ for a.e. $r \in (t_0,t)$, with $\|\Phi(r,\cdot)\|^2_{L^2} \leq C(t-r)^{-1/2}$, which is integrable in r over (t_0,t); note we are able to achieve a uniform bound for the second moment for $U(t,h)$ by considering the chaos expansion and the second moment bound given in (189). Since M_ε is an approximate identity, $\|\Phi(r,\cdot) * M_\varepsilon - \Phi(r,\cdot)\|_{L^2} \to 0$ for a.e. r as $\varepsilon \to 0$, and this quantity is dominated (uniformly in ε, using Young's inequality and $\|M_\varepsilon\|_{L^1} = 1$) by $2\|\Phi(r,\cdot)\|_{L^2}$, which is r-integrable as just shown. Dominated convergence gives $\int_{t_0}^{t} \|\Phi(r,\cdot) * M_\varepsilon - \Phi(r,\cdot)\|^2_{L^2}\,dr \to 0$.

$$\begin{aligned}&\int_{t_0}^{t}\int_{\mathbb{R}} K_{h_0}(r,y;t,h)U_\varepsilon(r,y)r^{-1/2}\overline{W}_\varepsilon(r,y)\,dy\,dr\\ &\qquad\longrightarrow \int_{t_0}^{t}\int_{\mathbb{R}} K_{h_0}(r,y;t,h)U(r,y)r^{-1/2}\,\overline{W}(dr,dy)\end{aligned} \tag{367}$$

in $L^2(W)$.

Step 3 We finally prove uniqueness Let U and V be two L^2-mild solutions satisfying (344), and set $D = U - V$. Since the equation is linear and the initial data cancel,

$$D(t,h) = \beta\int_{t_0}^{t}\int_{\mathbb{R}} K_{h_0}(r,y;t,h)D(r,y)r^{-1/2}\,\overline{W}(dr,dy).$$

By the Itô isometry,

$$\mathbb{E}|D(t,h)|^2 = \beta^2 \int_{t_0}^{t} \int_{\mathbb{R}} K_{h_0}(r,y;t,h)^2 r^{-1} \mathbb{E}|D(r,y)|^2 \, dy \, dr.$$

By Lemma 6.4,

$$\begin{aligned} \mathbb{E}|D(t,h)|^2 &= \beta^2 \int_{t_0}^{t} \int_{\mathbb{R}} K_{h_0}(r,y;t,h)^2 r^{-1} \, \mathbb{E}|D(r,y)|^2 \, dy \, dr \\ &\le \beta^2 C \, t_0^{-1} \int_{t_0}^{t} (t-r)^{-1/2} \, \mathbb{E}|D(r,h)|^2 \, dr, \end{aligned} \tag{368}$$

The integrability condition (344) ensures that these iterated integrals are finite. Iterating once more, applying Lemma 6.4 at each level and using $r_j \ge t_0$ to bound each r_j^{-1} factor by t_0^{-1}, we obtain after m iterations

$$\mathbb{E}\big[|D(t,h)|^2\big] \;\le\; \big(\beta^2 C t_0^{-1}\big)^m \int_{\Delta_m(t_0,t)} \prod_{j=0}^{m} (r_{j+1} - r_j)^{-1/2} \, d\mathbf{r}, \tag{369}$$

where $r_0 = t_0$, $r_{m+1} = t$, By the Dirichlet integral estimate, see Appendix D, we have

$$\int_{\Delta_m(t_0,t)} \prod_{j=0}^{m} (r_{j+1} - r_j)^{-1/2} \, d\mathbf{r} = (t - t_0)^{(m-1)/2} \frac{\Gamma(1/2)^{m+1}}{\Gamma((m+1)/2)}. \tag{370}$$

Combining with (369) and absorbing constants (depending only on β, C, t_0, T) into a single C',

$$\mathbb{E}\big[|D(t,h)|^2\big] \;\le\; \frac{(C')^m}{\Gamma((m+1)/2)}, \qquad m \ge 1. \tag{371}$$

Since the Gamma function grows super-exponentially,

$$\frac{(C')^m}{\Gamma((m+1)/2)} \longrightarrow 0, \qquad m \to \infty,$$

so that $\mathbb{E}\big[|D(t,h)|^2\big] = 0$. Since $t > t_0$ and $h \in \mathbb{R}$ were arbitrary, $D(t,h) = 0$ almost surely for every such (t,h), i.e. $U = V$ almost surely, proving uniqueness. Therefore

$$\mathbb{E}|D(t,h)|^2 = 0,$$

for every $t > t_0$ and $h \in \mathbb{R}$. Hence $U = V$ almost surely, which proves uniqueness.

Step 4 Now we can justify using Theorem 6.5 that the chaos expansion

$$U(t,h) = \sum_{k=0}^{\infty} \beta^k \int_{\Delta_k(t_0,t)} \int_{\mathbb{R}^k} \widehat{\rho}_k^{H;t,h}(\mathbf{r},\mathbf{y}) \prod_{j=1}^{k} \left(r_j^{-1/2} \, \overline{W}(dr_j, dy_j) \right), \tag{372}$$

can be considered the solution of the SPDE (343). The moment bounds established in (189) imply that $U(q,\zeta)$ satisfies the integrability condition needed in (344). Therefore, by Theorem 6.5, U is the unique L^2-mild solution of the SPDE (343). □

7 Tigthness of the Measure $P_{h_n}^{\mathrm{TILT}}$

Recall (**??**) that the tilted modified partition function $\mathfrak{Z}_n^{\mathrm{TILT}}(x,\beta)$ is given by

$$\mathfrak{Z}_n^{\mathrm{TILT}}(x,\beta) = E_{h_n}^{\mathrm{TILT}}\left[\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})\,\mathbb{1}_{\{S_n=x\}}\right]. \tag{373}$$

We are going to show that the continuous space–time process given by

$$(t,x) \mapsto \sqrt{n}\,\mathfrak{Z}_{nt}^{\mathrm{TILT}}(x\sqrt{n},\beta_n) \tag{374}$$

is tight as a random element of $C_b[(\epsilon,T)\times\mathbb{R}]$, where $0<\epsilon<T<\infty$. Our proof of tightness follows the strategy of [AKQ14b]. In that work, the modified partition function is shown to satisfy a discrete version of the parabolic Anderson model. Under diffusive scaling, this yields an integral representation, and standard SPDE arguments [Wal86, Kho14] then provide the required regularity estimates in space and time uniformly in n.

There is one additional technical point in our setting: the spatial domain is $\mathbb{R}$ is not compact. We therefore first restrict attention to compact spatial intervals $[-M,M]$, where the Arzelà–Ascoli theorem characterizes tightness; see, for example, [Bil99]. A standard diagonalisation argument then yields tightness in $C\big([\epsilon,T]\times\mathbb{R}\big)$ equipped with the topology of uniform convergence on compact sets. Finally, by obtaining uniform control of the tails, we upgrade this to tightness in $C_b\big([\epsilon,T]\times\mathbb{R}\big)$ equppied with the supremum norm. We break this into a four-step process for the convenience of the reader.

Step 1: Show that $\mathfrak{Z}^{\mathbf{TILT}}$ solves a discrete form of the parabolic Anderson model with drift, and show that under diffusive scaling we obtain an integral representation of $\mathfrak{Z}^{\mathbf{TILT}}$.

Step 2: Using this integral representation of $\mathfrak{Z}^{\mathbf{TILT}}$, obtain modulus of continuity estimates using a discrete form of the Burkholder-Davis-Gundy inequality.

Step 3: Use the GRR inequality (E.1), together with a diagonalization argument, to obtain tightness in $C([\epsilon,T]\times\mathbb{R})$ with the topology of uniform convergence on compact sets.

Step 4: Upgrade tightness to $C_b([\epsilon,T]\times\mathbb{R})$ by obtaining uniform tail bounds.

We remark that although the spatial domain is $\mathbb{R}$ is not compact, it is convenient to derive the integral estimates in Step 2 over the whole real line. The bounds obtained in this way are uniform and therefore remain valid when restricted to compact spatial sets.

Step 1 Consider the expectation under the tilted measure:

$$\begin{aligned}
&E_h^{\mathrm{TILT}}\left[\prod_{i=1}^{n+1}(1+\beta\omega_{i,S_i})\mathbb{1}_{\{S_{n+1}=x\}}\right] \\
&\quad= (1+\beta\omega_{n+1,x})\, E_h^{\mathrm{TILT}}\left[\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})\mathbb{1}_{\{S_{n+1}=x\}}\right] \\
&\quad= (1+\beta\omega_{n+1,x})\, E_h^{\mathrm{TILT}}\left[\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})E_h^{\mathrm{TILT}}\left[\mathbb{1}_{\{S_{n+1}=x\}} \mid \mathcal{F}_n\right]\right] \\
&\quad= (1+\beta\omega_{n+1,x})\, E_h^{\mathrm{TILT}}\left[\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})P_h^{\mathrm{TILT}}(S_{n+1}=x \mid \mathcal{F}_n)\right]. \qquad (375)
\end{aligned}$$

where for the second equality, we use the tower property with respect to the standard SRW filtration $\mathcal{F}_n$. By the Markov property and the fact that a nearest-neighbor walk must be at $x \pm 1$ at time n, to reach x at time $n+1$, we have:

$$\begin{aligned}
P_h^{\mathrm{TILT}}(S_{n+1}=x \mid \mathcal{F}_n) &= P_h^{\mathrm{TILT}}(S_{n+1}=x \mid S_n) \\
&= P_h^{\mathrm{TILT}}\big(S_{n+1}=x \mid S_n=x-1\big)\, \mathbb{1}_{\{S_n=x-1\}} \\
&\quad + P_h^{\mathrm{TILT}}\big(S_{n+1}=x \mid S_n=x+1\big)\, \mathbb{1}_{\{S_n=x+1\}}. \qquad (376)
\end{aligned}$$

Substituting this back into (375), we obtain:

$$\begin{aligned}
&(1+\beta\omega_{n+1,x})E_h^{\mathrm{TILT}}\left[\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})\, \mathbb{1}_{\{S_{n+1}=x\}}\right] \\
&\quad= (1+\beta\omega_{n+1,x})E_h^{\mathrm{TILT}}\left[\prod_{i=1}^{n}(1+\beta\omega_{i,S_i})\Big(p_h\mathbb{1}_{\{S_n=x-1\}} + q_h\mathbb{1}_{\{S_n=x+1\}}\Big)\right]. \qquad (377)
\end{aligned}$$

where the tilted transition probabilities are defined as:

$$p_h := \frac{e^h}{2\cosh h}, \qquad q_h := \frac{e^{-h}}{2\cosh h}. \qquad (378)$$

This allows us to write the recursion for the tilted partition function $\mathfrak{Z}_n^{\mathrm{TILT}}(x)$ as:

$$\mathfrak{Z}_{n+1}^{\mathrm{TILT}}(x) = (1+\beta\omega_{n+1,x})\Big[p_h\mathfrak{Z}_n^{\mathrm{TILT}}(x-1) + q_h\mathfrak{Z}_n^{\mathrm{TILT}}(x+1)\Big]. \qquad (379)$$

Subtracting $\mathfrak{Z}_n^{\mathrm{TILT}}(x)$ from both sides for (379), we have,

$$\mathfrak{Z}_{n+1}^{\mathrm{TILT}}(x) - \mathfrak{Z}_n^{\mathrm{TILT}}(x) = \frac{1}{2}\Delta \mathfrak{Z}_n^{\mathrm{TILT}}(x) + (q_h - p_h)\nabla \mathfrak{Z}_n^{\mathrm{TILT}}(x) + \beta\omega_{n+1,x}\big[p_h \mathfrak{Z}_n^{\mathrm{TILT}}(x-1) + q_h \mathfrak{Z}_n^{\mathrm{TILT}}(x+1)\big], \quad (380)$$

where the discrete Laplacian Δ and centered difference ∇ are:

$$\Delta \mathfrak{Z}_n^{\mathrm{TILT}}(x) := \mathfrak{Z}_n^{\mathrm{TILT}}(x+1) - 2\mathfrak{Z}_n^{\mathrm{TILT}}(x) + \mathfrak{Z}_n^{\mathrm{TILT}}(x-1), \quad (381)$$

$$\nabla \mathfrak{Z}_n^{\mathrm{TILT}}(x) := \frac{\mathfrak{Z}_n^{\mathrm{TILT}}(x+1) - \mathfrak{Z}_n^{\mathrm{TILT}}(x-1)}{2}. \quad (382)$$

Since $q_h - p_h = \tanh h$, the drift term becomes explicitly visible, yielding the discrete SPDE:

$$\mathfrak{Z}_{n+1}^{\mathrm{TILT}}(x) - \mathfrak{Z}_n^{\mathrm{TILT}}(x) = \frac{1}{2}\Delta \mathfrak{Z}_n^{\mathrm{TILT}}(x) - \tanh h\, \nabla \mathfrak{Z}_n^{\mathrm{TILT}}(x) + W_{n+1}. \quad (383)$$

where

$$W_{n+1}(x) = \beta\omega_{n+1,x}\Big[p_h \mathfrak{Z}_n^{\mathrm{TILT}}(x-1) + q_h \mathfrak{Z}_n^{\mathrm{TILT}}(x+1)\Big] \quad (384)$$

We define the one-step transition operator G_h on a function $\phi(x)$ by

$$(G_h\phi)(x) := p_h\phi(x-1) + q_h\phi(x+1), \quad (385)$$

Using this operator, the recursion (379) can be expressed as

$$\mathfrak{Z}_{n+1}^{\mathrm{TILT}}(x) = \big(1 + \beta\omega_{n+1,x}\big)(G_h \mathfrak{Z}_n^{\mathrm{TILT}})(x). \quad (386)$$

Let $\mathfrak{p}_h^n(x,y)$ denote the n-step transition kernel (Green's function) of the tilted random walk:

$$\mathfrak{p}_h^n(x,y) := P_h^{\mathrm{TILT}}(S_n = x \mid S_0 = y)\,. \quad (387)$$

The action of the n-th power of the operator G_h is given by the convolution

$$(G_h^n f)(x) = \sum_{y\in\mathbb{Z}} \mathfrak{p}_h^n(x,y) f(y). \quad (388)$$

NOTE: I am tired now, but looks correct, I need to double everything **NOTE:** say something about the intaiil data

Iterating the recursion (386) and applying the superposition principle (see [Law10]), we obtain the discrete mild form for the tilted partition function:

$$\mathfrak{Z}_n^{\mathrm{TILT}}(x) = \sum_{y\in\mathbb{Z}} \mathfrak{p}_h^n(x,y)\mathfrak{Z}_0(y) + \beta_n \sum_{k=1}^{n}\sum_{y\in\mathbb{Z}} \mathfrak{p}_h^{n-k}(x,y)\,\omega_{k,y}\,(G_h \mathfrak{Z}_{k-1}^{\mathrm{TILT}})(y). \quad (389)$$

We assume a flat initial condition $\mathfrak{z}_0(y) = 1$ for all $y \in \mathbb{Z}$, which implies that the first term in (391) is identically 1, since $\mathfrak{p}_h^n$ is a probability transition kernel. By choosing the Dirac initial condition $\mathfrak{z}_0(y) = \delta_0(y)$, the first term of the discrete mild form represents the transition probability of the underlying random walk. We define the rescaled field

$$z_n(t,x) = n^{1/2}\mathfrak{z}^{\mathrm{TILT}}_{\lfloor nt \rfloor}(\lfloor x\sqrt{n} \rfloor), \tag{390}$$

and rewrite the lattice recursion as a spacetime Riemann sum:

$$z_n(t,x) = p_n^{\mathrm{TILT}}(t,x) + \beta_n \int_0^t \int_{\mathbb{R}} p_n^{\mathrm{TILT}}(t-s, x-y)\overline{z}_n(s,y) w_n\left(s + \frac{1}{n}, y\right) \mu_n(ds, dy), \tag{391}$$

NOTE: need to explain what w_n is where

$$p_n^{\mathrm{TILT}}(t,x) = \sqrt{n}\mathfrak{p}_h^{\lfloor nt \rfloor}(\lfloor x\sqrt{n} \rfloor, 0) = \sqrt{n} P^{\mathrm{TILT}}_{\lfloor nt \rfloor}(\lfloor x\sqrt{n} \rfloor)$$

and

$$\overline{z}_n(s,y) := n^{1/2}\left(G_h \mathfrak{z}^{\mathrm{TILT}}_{\lfloor ns \rfloor}\right)(\lfloor y\sqrt{n} \rfloor). \tag{392}$$

and the discrete measure μ_n on the scaled spacetime lattice

$$\mathbb{T}_n = (n^{-1}\mathbb{Z} \times n^{-1/2}\mathbb{Z})$$

is defined by

$$\mu_n(A) = n^{-3/2} \cdot \#(A \cap \mathbb{T}_n). \tag{393}$$

The scaling $\beta_n = \beta n^{-1/4}$ places the system in the intermediate disorder regime. Under the diffusive rescaling $z_n \sim n^{1/2}$, the variance of the integrated noise term over a unit spacetime region of size $n^{-3/2}$ is of order n^{-1}, which matches the scaling of spacetime white noise in $1+1$ dimensions.

By extending the w_n and $\overline{z}_n$ to be piecewise constant on the lattice rectangles **NOTE:** Seems sus her

$$B_{k,j} = [k/n, (k+1)/n) \times [j/\sqrt{n}, (j+1)/\sqrt{n}),$$

The spacetime sums become Riemann sums with respect to μ_n, which converge to Lebesgue integrals. This allows the evolution to be rewritten as

$$z_n(t,x) = p_n^{\mathrm{TILT}}(t,x) + \beta \int_0^t \int_{\mathbb{R}} p_n^{\mathrm{TILT}}(t-s, x-y)\overline{z}_n(s,y) w_n\left(s + \frac{1}{n}, y\right) dy\, ds \tag{394}$$

Step 2 Now that we have obtained an integral representation of (373), we turn to modulus of continuity estimates. With initial data $\mathfrak{z}_0(y) = \epsilon$, for some $0 <$

$\epsilon < t$, the representation in (394) naturally splits into two terms, Specifically, we consider

$$A_{n,\epsilon}(x,t) = \int_{\epsilon}^{t} p_n^{\mathrm{TILT}}(t,x).dt \tag{395}$$

$$M_{n,\epsilon}(x,t) := \beta \int_{\epsilon}^{t} \int_{\mathbb{R}} p_n^{\mathrm{TILT}}(t-s,x-y)\, \overline{z}_n(s,y)\, w_n\left(s+\frac{1}{n},y\right)\, dy\, ds. \tag{396}$$

To apply the GRR inequality (E.1), it suffices to derive suitable moment bounds for (395) and (396). The M-th moment of the term in (395) can be bounded by a constant C depending on t, ϵ, and M. More precisely,

$$\mathbb{E}\big[\,|A_{n,\epsilon}(x,t)|^M\big] \leq C. \tag{397}$$

where we have used (A.2) along with Remark (3). For (396) our main tool will be a discrete version of the Burkholder–Davis–Gundy inequality. We first show that (396) defines a martingale, and then obtain its quadratic variation.

Lemma 7.1. *Let*

$$M_{n,\varepsilon}(x,t) := \int_{\varepsilon}^{t} \int_{\mathbb{R}} p_n^{\mathrm{TILT}}(t-s,x-y)\, \overline{z}_n(s,y)\ \omega\left(s+\frac{1}{n},y\right)\, dy\, ds. \tag{398}$$

Then, for each fixed $x \in \mathbb{R}$, the process $\big(M_{n,\varepsilon}(x,t)\big)_{t\geq\varepsilon}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq\epsilon}$ where

$$\mathcal{F}_t := \sigma(\omega(s,y) :\ s \leq \lfloor nt \rfloor,\ y \in \mathbb{R}). \tag{399}$$

Moreover, its quadratic variation is given by

$$\langle M_{n,\varepsilon}(\cdot,x)\rangle_t = \int_{\varepsilon}^{t} \int_{\mathbb{R}} \big(p_n^{\mathrm{TILT}}(t-s,x-y)\, \overline{z}_n(s,y)\big)^2\, dy\, ds. \tag{400}$$

Proof. Fix $x \in \mathbb{R}$ and let $\varepsilon \leq \tilde{t} < t$. To show that $M_{n,\varepsilon}(x,t)$ is a martingale, it is enough to prove that

$$\mathbb{E}\big[M_{n,\varepsilon}(x,t) \mid \mathcal{F}_{\tilde{t}}\big] = M_{n,\varepsilon}(x,\tilde{t}).$$

We decompose

$$\begin{aligned} M_{n,\varepsilon}(x,t) = &\int_{\varepsilon}^{\tilde{t}} \int_{\mathbb{R}} p_n^{\mathrm{TILT}}(t-s,x-y)\, \overline{z}_n(s,y)\, \omega\left(s+\frac{1}{n},y\right)\, dy\, ds \\ &+ \int_{\tilde{t}}^{t} \int_{\mathbb{R}} p_n^{\mathrm{TILT}}(t-s,x-y)\, \overline{z}_n(s,y)\, \omega\left(s+\frac{1}{n},y\right)\, dy\, ds. \end{aligned}$$

The definition of $\overline{z}_n(s,y)$ in (392), the process $\overline{z}_n(s,y)$ is $\mathcal{F}_s$-measurable, and hence $\mathcal{F}_{\tilde{t}}$-measurable for $s \le \tilde{t}$. Since the environment is centred and independent of the past, the contribution from the interval $[\tilde{t}, t]$ has conditional expectation zero. Therefore, the second term has conditional expectation 0. Hence

$$\mathbb{E}\big[M_{n,\varepsilon}(x,t) \mid \mathcal{F}_{\tilde{t}}\big] = \int_{\varepsilon}^{\tilde{t}} \int_{\mathbb{R}} p_n^{\mathrm{TILT}}(t-s, x-y)\, \overline{z}_n(s,y)\, \omega\left(s + \frac{1}{n}, y\right)\, dy\, ds.$$

To calculate the quadratic variation, we use the orthogonality of the noise and the fact that the environment has variance one. This yields

$$\begin{aligned}
\langle M_{n,\varepsilon}(\cdot,x)\rangle_t &= \int_{\varepsilon}^{t} \int_{\mathbb{R}} \big(p_n^{\mathrm{TILT}}(t-s,x-y)\, \overline{z}_n(s,y)\big)^2 \mathbb{E}\left[\omega\left(s+\frac{1}{n}, y\right)^2\right] dy\, ds \\
&= \int_{\varepsilon}^{t} \int_{\mathbb{R}} \big(p_n^{\mathrm{TILT}}(t-s,x-y)\, \overline{z}_n(s,y)\big)^2\, dy\, ds,
\end{aligned}$$

which proves the claim. □

Now we can simply apply the Burkholder-Davis-Gundy Inequality. The need for $M > 6$ will be required in our GRR estimates. **NOTE:** overfill

Theorem 7.2. *Let $M > 6$. Then there exists a constant $C_M > 0$ such that*

$$\begin{aligned}
&\mathbb{E}\Big[\big|M_{n,\varepsilon}(x,t+\delta) - M_{n,\varepsilon}(x,t)\big|^M\Big] \\
&\qquad \le C_M\, \mathbb{E}\left[\left(\int_{\varepsilon}^{t} \int_{\mathbb{R}} \big(p_n^{\mathrm{TILT}}(t+\delta-s,x-y) - p_n^{\mathrm{TILT}}(t-s,x-y)\big)^2 \big(\overline{z}_n(s,y)\big)^2\, dy\, ds\right)^{M/2}\right],
\end{aligned} \tag{401}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\big|M_{n,\varepsilon}(x+\delta,t) - M_{n,\varepsilon}(x,t)\big|^M\Big] \\
&\qquad \le C_M\, \mathbb{E}\left[\left(\int_{\varepsilon}^{t} \int_{\mathbb{R}} \big(p_n^{\mathrm{TILT}}(t-s,x+\delta-y) - p_n^{\mathrm{TILT}}(t-s,x-y)\big)^2 \big(\overline{z}_n(s,y)\big)^2\, dy\, ds\right)^{M/2}\right]
\end{aligned} \tag{402}$$

Applying Hölder's inequality to (402) with exponents $p = M/2$ and $q = M/(M-$

2) gives

$$
\begin{aligned}
&\mathbb{E}\Big[\big|M_{n,\varepsilon}(x+\delta,t)-M_{n,\varepsilon}(x,t)\big|^{M}\Big] \\
&\le C'_M\left(\int_{\varepsilon}^{t}\int_{\mathbb{R}}\big|p_n(t-s,x+\delta-y)-p_n(t-s,x-y)\big|^{\frac{2M}{M-2}}\,dy\,ds\right)^{\frac{M-2}{2}} \qquad (403)\\
&\qquad\times\int_{\varepsilon}^{t}\int_{\mathbb{R}}\mathbb{E}\big[\overline{z}_n(s,y))\big|^{M}\big]\,dy\,ds. \qquad (404)
\end{aligned}
$$

We provide an estimate for (404), which is based on [? , Lemma 3.1]. From (??) we have that

$$
\mathbb{E}\big[\,|\overline{z}_n(s,y)|^{M}\big] \le \big(p_n^{\mathrm{TILT}}(s,y)\big)^{2M}. \qquad (405)
$$

Then, using (A.2), we can bound $\big(p_n^{\mathrm{TILT}}(s,y)\big)^{2M}$ by a shifted heat kernel (see Remark (3)), which, once integrated over space and time from ϵ to t, produces a constant depending on M, ϵ, and t.

We proceed to estimate (411) using (A.2). Following Remark (3) and (A.3) it is good enough to bound the difference of the discrete kernels and provide the estimates using the continuous version. In this spirit we define

$$
\rho^{\mathrm{TILT}}(t-s,x-y,h)=\frac{C}{\sqrt{t-s}}\exp\!\left(-c\,\frac{(x-y-h(t-s))^2}{t-s}\right). \qquad (406)
$$

Lemma 7.3. *Let $0<\delta<1$ and fix t,s,h. Define $\Delta_x(t-s,x-y,h):=\rho^{TILT}(t-s,x-y+\delta,h)-\rho^{TILT}(t-s,x-y,h)$. Then for each even $M>6$ we have*

$$
\left(\int_{\epsilon}^{t}\int_{\mathbb{R}}\Delta_x(t-s,x-y,h)^{\frac{2M}{M-2}}\,dy\,ds\right)^{\frac{M}{2}-1}\le C\,\delta^{\frac{M}{2}-1}, \qquad (407)
$$

where C is a constant depending on t, ϵ, and M.

Proof. Note that for each $M>6$ we have that $1\le M/(M-2)\le 2$. Using (E.3) with $K=\frac{M}{M-2}$ and using Fourier Inversion we obtain. $\frac{2K}{2K-1}=\frac{2M}{M+2}$, we obtain

$$
\int_{-\infty}^{\infty}\big|\Delta_x(t-s,x-y,h)\big|^{\frac{2M}{M-2}}\,dy\lesssim\left(\int_{\mathbb{R}}\big|e^{-i\xi\delta}-1\big|^{\frac{2M}{M+2}}\exp\!\Big(-\tilde{K}(t-s)\xi^2\Big)\,d\xi\right)^{\frac{M+2}{M-2}}, \qquad (408)
$$

where $\tilde{K}:=\tilde{K}(M)=M/4(M+2)$. An application of Fubini's Theorem, followed by integration over the time variable s, yields:

$$
\begin{aligned}
&\int_\epsilon^t \int_{-\infty}^{\infty} \exp\left(-\tilde{K}(t-s)\xi^2\right) \left|1-e^{i\xi\delta}\right|^{\frac{2M}{M+2}}\, d\xi\, ds \\
&\quad = \int_{-\infty}^{\infty} \left[\int_\epsilon^t \exp\left(-\tilde{K}(t-s)\xi^2\right)\, ds\right] \left|1-e^{i\xi\delta}\right|^{\frac{2M}{M+2}}\, d\xi \\
&\quad = \int_{-\infty}^{\infty} \frac{1-\exp\left(-\tilde{K}(t-\epsilon)\xi^2\right)}{\tilde{K}\xi^2} \left|1-e^{i\xi\delta}\right|^{\frac{2M}{M+2}}\, d\xi \\
&\quad = \frac{1}{2}\int_0^{\infty} \frac{1-\exp\left(-\tilde{K}(t-\epsilon)\xi^2\right)}{\xi^2} \left|1-e^{i\xi\delta}\right|^{\frac{2M}{M+2}}\, d\xi
\end{aligned} \tag{409}
$$

Using the bounds $\left|1-e^{i\xi\delta}\right| \leq \min\bigl(1, |\xi\delta|\bigr)$ and $\left|1-e^{-|\theta|}\right| \leq 1$ for all $\theta \in \mathbb{R}$, we obtain from (409) that

$$
\int_0^{\infty} \frac{1-\exp\bigl(-K(t-\epsilon)\xi^2\bigr)}{\xi^2} \left|1-e^{i\xi\delta}\right|^{\frac{2M}{M+2}}\, d\xi \lesssim \int_0^{\infty} \frac{\min\bigl(1, |\xi\delta|^{\frac{2M}{M+2}}\bigr)}{\xi^2}\, d\xi.
$$

Splitting this into the regions $\xi\delta \leq 1$ and $\xi\delta \geq 1$, we obtain

$$
\int_0^{\infty} \frac{\min\bigl(1, |\xi\delta|^{\frac{2M}{M+2}}\bigr)}{\xi^2}\, d\xi \lesssim \int_0^{1/\delta} \delta^{\frac{2M}{M+2}} \xi^{-\frac{2}{M+2}}\, d\xi \;+\; \int_{1/\delta}^{\infty} \frac{1}{\xi^2}\, d\xi.
$$

Therefore,

$$
\left(\int_0^{1/\delta} \delta^{\frac{2M}{M+2}} \xi^{-\frac{2}{M+2}}\, d\xi \;+\; \int_{1/\delta}^{\infty} \frac{1}{\xi^2}\, d\xi\right)^{\frac{M+2}{M-2}} \lesssim \delta^{\frac{M+2}{M-2}} \lesssim \delta,
$$

where the last inequality follows since $\delta < 1$ and $\frac{M+2}{M-2} > 1$. □

We now proceed to estimate

$$
\mathbb{E}\Bigl[\bigl|M_{n,\varepsilon}(x+\delta, t) - M_{n,\varepsilon}(x,t)\bigr|^M\Bigr] \tag{410}
$$

Applying Hölder's inequality to (402) with exponents $p = M/2$ and $q = M/(M-2)$ gives

$$
\begin{aligned}
&\mathbb{E}\Bigl[\bigl|M_{n,\varepsilon}(x+\delta, t) - M_{n,\varepsilon}(x,t)\bigr|^M\Bigr] \\
&\leq C'_M \left(\int_\varepsilon^t \int_{\mathbb{R}} \bigl|p_n(t+\delta-s, x-y) - p_n(t-s, x-y)\bigr|^{\frac{2M}{M-2}}\, dy\, ds\right)^{\frac{M-2}{2}} && (411) \\
&\qquad \times \int_\varepsilon^t \int_{\mathbb{R}} \mathbb{E}\bigl[|\overline{z}_n(s,y)|^M\bigr]\, dy\, ds. && (412)
\end{aligned}
$$

We can provide bounds (412) using (D.1), as discussed in **NOTE:** reference fail?. We proceed to estimate (411) using (A.2) and Remark (3) in the same spirit as (9). Recall the defintion of $\rho^{\mathrm{TILT}}(t-s+\delta, x-y, h)$ from (406).

Lemma 7.4. ***NOTE:*** *Check Calculation Let $0<\delta<1$. Define $\Delta_t(t-s, x-y, h) := \rho^{\mathrm{TILT}}(t-s+\delta, x-y, h) - \rho^{\mathrm{TILT}}(t-s, x-y, h)$. Then for each even integer $M>6$, we have*

$$\left(\int_\epsilon^t \int_{\mathbb{R}} |\Delta_t(t-s, x-y, h)|^{\frac{2M}{M-2}}\, dy\, ds\right)^{\frac{M}{2}-1} \le C\delta^{\frac{M}{4}-2}.$$

where C is a constant depending on t, ϵ, and M.

Proof. Again using (E.3) with $K = \frac{M}{M-2}$ and $\frac{2K}{2K-1} = \frac{2M}{M+2}$, we obtain

$$\int_{-\infty}^{\infty} |\Delta_t(t-s, x-y, h)|^{\frac{2M}{M-2}}\, dy \lesssim \left(\int_{-\infty}^{\infty} \left|\exp\left(-\frac{(t-s+\delta)\xi^2}{8}\right) - \exp\left(-\frac{(t-s)\xi^2}{8}\right)\right|^{\frac{2M}{M+2}} d\xi\right)^{\frac{M+2}{M-2}} \tag{413}$$

$$= \left(\int_{-\infty}^{\infty} \exp\left(-\frac{M(t-s)\xi^2}{4(M+2)}\right) \left|1-\exp\left(-\frac{\delta\xi^2}{8}\right)\right|^{\frac{2M}{M+2}} d\xi\right)^{\frac{M+2}{M-2}}. \tag{414}$$

Let $\tilde{K} := \tilde{K}(M) = \frac{M}{4(M+2)}$. and using Fubini's theorem,

$$\begin{aligned}
&\int_\epsilon^t \int_{-\infty}^{\infty} \exp\left(-\tilde{K}(t-s)\xi^2\right) \left|1-\exp\left(-\frac{\delta\xi^2}{8}\right)\right|^{\frac{2M}{M+2}} d\xi\, ds \\
&= \int_{-\infty}^{\infty} \left[\int_\epsilon^t \exp\left(-\tilde{K}(t-s)\xi^2\right) ds\right] \left|1-\exp\left(-\frac{\delta\xi^2}{8}\right)\right|^{\frac{2M}{M+2}} d\xi \\
&= \int_{-\infty}^{\infty} \frac{1-\exp\left(-\tilde{K}(t-\epsilon)\xi^2\right)}{\tilde{K}\xi^2} \left|1-\exp\left(-\frac{\delta\xi^2}{8}\right)\right|^{\frac{2M}{M+2}} d\xi.
\end{aligned} \tag{415}$$

Using the bound $|e^{-u}-1| \le u$ and $u \ge 0$, we have

$$(415) \lesssim \int_0^\infty \frac{\left|1-e^{-\delta\xi^2/8}\right|^{\frac{2M}{M+2}}}{\xi^2}\, d\xi$$

Make the change of variables $u = \sqrt{\delta}\,\xi$ so that

$$\int_0^\infty \frac{\left|1-e^{-\delta\xi^2/8}\right|^{\frac{2M}{M+2}}}{\xi^2}\, d\xi = \sqrt{\delta}\int_0^\infty \frac{\left|1-e^{-u^2/8}\right|^{\frac{2M}{M+2}}}{u^2}\, du \lesssim \sqrt{\delta}.$$

Therefore,

$$\left(\int_0^\infty \frac{\left|1-e^{-\delta\xi^2/8}\right|^{\frac{2M}{M+2}}}{\xi^2}\,d\xi\right)^{\frac{M+2}{M-2}} \lesssim \delta^{\frac{M+2}{2(M-2)}} \lesssim \delta^{1/2},$$

where the last inequality follows since $\delta < 1$ and $\frac{M+2}{(M-2)} > 1$ for $M \geq 6$. Raising to the power of $M/2-1$ gives the required result. □

Using (7.3) and (7.4) we are able to conclude that

Lemma 7.5. *Let $0 \leq \delta_1, \delta_2 \leq 1$. For each even $M > 6$ and each $\varepsilon > 0$, there exists a constant $C = C(M, \varepsilon, t)$ such that*

$$Q\Big[\big|\overline{z}_n(s,y) - \overline{z}_n(s+\delta_1, y+\delta_2)\big|^M\Big] \leq C\big(|\delta_1| + |\delta_2|\big)^{\frac{1}{4}-\frac{2}{M}}. \tag{416}$$

Step 3 To apply the Garsia–Rodemich–Rumsey inequality , it suffices to verify that for an $\alpha > 0$,

$$\int\int_R \frac{\mathbb{E}\left[|z_n(x,t) - z_n(y,s)|^M\right]}{d(s,t,x,y)^{\alpha M}}\,ds\,dt\,dx\,dy < \infty,$$

where $d(s,t,x,y) = \sqrt{(t-s)^2 + (x-y)^2}$ and $R = \{(s,t,x,y) \in [\epsilon, T]^2 \times \mathbb{R}^2\}$. Using the moment bound (416), the integrand behaves like $d(s,t,x,y)^{M/2-\alpha M}$, which is integrable over R provided $\alpha < 1/4$. Therefore, by the Garsia–Rodemich–Rumsey inequality (E.1), we have

$$|z_n(x,t) - z_n(y,s)| \leq C_{M,\epsilon}\left((t-s)^2 + (x-y)^2\right)^{\alpha/2} \tag{417}$$

Q almost surely. This implies the sequence $z_n(x,t)$ is equicontinuous Q almost surely on R. **NOTE:** do I need a diagnolization argument?. **NOTE:** Should be finished by Arzera Ascoli.

Step 4

8 Tightness of the Measure for P_h^{AREA}

9 Tightness For The Velocity Partition Function

We recall the definition of the velocity partition function from (228).

$$\mathbf{E}_{t,h}^H\left[\prod_{i=1}^n e^{\beta_n \omega(i,S_i) - \lambda(\beta_n)} 1_{\{S_n = x\}}\right] \tag{418}$$

Our proof of tightness will follow the exact same steps as outlines in Section 7, and we will make the necessary modifications.

We will show that the continuously interpolated space–time process

$$(t,h) \longmapsto z_n^H(t,h) := \sqrt{n}\, \mathfrak{Z}^{\text{Velc}}_{\lfloor nt \rfloor}\left(\left[\sqrt{n}\, t(h-h_0)\right]_{\lfloor nt \rfloor}, \beta_n \right) \tag{419}$$

is tight as a random element of

$$C\big([t_0+\epsilon, T]\times \mathbb{R}\big),$$

where $\epsilon > 0$ and $t_0 + \epsilon < T < \infty$.

The main idea **NOTE:** informal is that the modified velocity partition function has an integral representation, following a similar interpolation scheme as outlines in . For $t_0 \le t \le T$, this representation is given by

$$\begin{aligned} z_n^H(t,h) &= K_{h_0,n}(t_0,h_0;t,h) \\ &\quad + \beta \int_{t_0}^t \int_{\mathbb{R}} K_{h_0,n}(s,\eta;t,h)\, \overline{z}_n^H(s,\eta)\, w_n^H\left(s+\frac{1}{n},\eta\right) d\eta\, ds. \end{aligned} \tag{420}$$

where we define the discrete velocity kernel by

$$K_{h_0,n}(s,\eta;t,h) := t\, p_n\big(\lfloor ns \rfloor, \lfloor \sqrt{n}\, s(\eta-h_0)\rfloor; \lfloor nt \rfloor, \lfloor \sqrt{n}\, t(h-h_0)\rfloor\big)\,, \tag{421}$$

and where $p_n(x,t) = \sqrt{n} p(nt, [x\sqrt{n}]_{nt})$, which is the standard rescaled SRW transition probabilities. **NOTE:** What exactly is this? from **NOTE:** this needs to be discrete **NOTE:** dont like this. We also define $\overline{z}_n^H(s,\eta)$ in the same way as (392), using the same n^{th} power of the operator G_h with the n-step transition kernel of the discrete diffusion H^i as defined in (174) . In the same manner as Theorem 7.1 one can check that discrete stochastic convolution in $z_n^H(t,h)$ as defined in (420) is a martingale with respect to the filtration which is given by $\mathcal{F}_t := \sigma(\omega(s,y):\ s \le \lfloor nt \rfloor,\ y \in \mathbb{R})$. Further one can verify that it's quadratic varaition is given by

$$\langle M_{n,\varepsilon}(\cdot,x)\rangle_t = \int_{t_0+\varepsilon}^t \int_{\mathbb{R}} \big(K_{h_0,n}(s,\eta;t,h)\, \overline{z}_n(s,y)\big)^2\, d\eta\, ds. \tag{422}$$

Now let $|h-\tilde{h}| \le 1$ and $|t-\tilde{t}| \le 1$. Therefore by an application of the Burkholder Davis Gundy Inequality we have that

$$\begin{aligned} &\mathbb{E}\left[|M_{n,\varepsilon}(t,h) - M_{n,\varepsilon}(\tilde{t},h)|^M\right] \\ &\quad \le C_M \mathbb{E}\left[\left(\int_{t_0+\varepsilon}^t \int_{\mathbb{R}} \big(K_{h_0,n}(s,\eta;t,h) - K_{h_0,n}(s,\eta;\tilde{t},h)\big)^2 \big(\overline{z}_n^H(s,\eta)\big)^2\, d\eta\, ds\right)^{M/2}\right], \end{aligned} \tag{423}$$

and

$$
\begin{aligned}
&\mathbb{E}\left[|M_{n,\varepsilon}(t,\tilde{h})-M_{n,\varepsilon}(t,h)|^M\right]\\
&\quad\le C_M\mathbb{E}\left[\left(\int_{t_0+\varepsilon}^t\int_{\mathbb{R}}\Big(K_{h_0,n}(s,\eta;t,\tilde{h})-K_{h_0,n}(s,\eta;t,h)\Big)^2\big(\overline{z}_n^H(s,\eta)\big)^2\,d\eta\,ds\right)^{M/2}\right].
\end{aligned}
\tag{424}
$$

Applying Hölder's inequality with conjugate exponents $\frac{M}{M-2}$ and $\frac{M}{2}$ to (424) and (423), we reduce the increment estimates to suitable $L^{\frac{2M}{M-2}}$-bounds on the kernel differences, together with a uniform M-th moment bound for $\overline{z}_n^H$. More precisely, we use

$$
\sup_{n\ge 1}\sup_{(s,\eta)\in[t_0+\epsilon,T]\times\mathbb{R}}\mathbb{E}\left[\left|\overline{z}_n^H(s,\eta)\right|^M\right]<\infty. \tag{425}
$$

The proof of (425) follows the same Volterra-inequality argument as in Appendix D. The only modification is that the underlying transition kernel is now that of the discrete diffusion defined in (174). Since this diffusion is obtained from the simple random walk by a deterministic space–time change of variables, and since each t_i lies in the fixed compact interval $[t_0+\epsilon,T]$, the corresponding kernel bounds are uniform in n. Hence the argument of Appendix D applies without further change.

What is needed is the necessary modification of Theorem 7.4 and Theorem 7.3 for the kernel $K_{h_0,n}(s,\eta;t,h)$. We encounter the same issue as in the discussion preceding (406): the estimates required for tightness are formulated for the discrete kernel, whereas the available difference bounds are proved for its continuous counterpart. In the present setting, this comparison is justified by (174), which shows that the discrete diffusion may be considered as the SRW with a deterministic space–time change of variables. Since the times t_i remain in a fixed compact (t_0,T), the corresponding change-of-variable factors are uniformly bounded above and below. Therefore the discrete-to- continuous comparison in (3), together with (A.3), produces the required discrete spatial and time difference estimates uniformly in n. Thus, recalling the argument in (406), it is satisfactory to use the corresponding bounds for the continuous kernel. This leads us to;

Lemma 9.1. *For each even $M>6$ such that $|h-\tilde{h}|\le 1$*

$$
\left(\int_{t_0+\epsilon}^t\int_{\mathbb{R}}|K_{h_0}(s,\eta;t,h)-K_{h_0}|(s,\eta;t,\tilde{h})^{\frac{2M}{M-2}}\,dy\,ds\right)^{\frac{M}{2}-1} \tag{426}
$$

$$
\le C\,|h-\tilde{h}|^{\frac{M}{2}-1}, \tag{427}
$$

where C is a constant depending on t,T,ϵ and M.

Proof. Recall the definition of $K_{h_0}(s,\eta;t,h)$ from (264):

$$K_{h_0}(s,\eta;t,h) = \frac{t}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{\big(th - s\eta - h_0(t-s)\big)^2}{2(t-s)}\right\}. \tag{428}$$

Recall the standard heat kernel given by

$$\rho(t-s, x-y) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{t-s}}. \tag{429}$$

Then note that

$$t\rho\big(t-s, t(h-h_0) - s(\eta-h_0)\big) = K_{h_0}(s,\eta;t,h). \tag{430}$$

For fixed t and varying h, we have that

$$\begin{aligned} &|K_{h_0}(s,\eta;t,h) - K_{h_0}(s,\eta;t,\tilde{h})| \\ &\quad= \big|t\rho\big(t-s, t(h-h_0) - s(\eta-h_0)\big) \\ &\qquad\quad - t\rho\big(t-s, t(\tilde{h}-h_0) - s(\eta-h_0)\big)\big| \qquad (431)\\ &\quad\leq T\left|\rho\big(t-s, t(h-h_0) - s(\eta-h_0)\big) - \rho\big(t-s, t(\tilde{h}-h_0) - s(\eta-h_0)\big)\right| \qquad (432) \end{aligned}$$

where the last line follows since $t \leq T$. The result then directly follows from an application of (9) as $\big(t(h-h_0)-s(\eta-h_0)\big) - \big(t(\tilde{h}-h_0)-s(\eta-h_0)\big) = t(h-\tilde{h})$. □

Now we look at the fixing h and varying time t.

Lemma 9.2. *For each even $M > 6$ such that $|t-\tilde{t}| \leq 1$, then*

$$\begin{aligned} &\left(\int_{t_0+\epsilon}^{t}\int_{\mathbb{R}} |K_{h_0}(s,\eta;t,h) - K_{h_0}(s,\eta;\tilde{t},h)|^{\frac{2M}{M-2}}\,dy\,ds\right)^{\frac{M}{2}-1} \\ &\qquad\qquad \leq C\left(|t-\tilde{t}|^{\frac{M}{2}-1} + |t-\tilde{t}|^{\frac{M}{2}-1} + |t-\tilde{t}|^{\frac{M}{4}-2}\right) \end{aligned} \tag{433}$$

Proof. Fixing h and varying time t, such that $|t-\tilde{t}| \leq 1$ we obtain that

$$\begin{aligned} &|K_{h_0}(s,\eta;t,h) - K_{h_0}(s,\eta;t,\tilde{h})| \\ &\quad= \left|\begin{matrix} t\rho\big(t-s, t(h-h_0) - s(\eta-h_0)\big) \\ - t\rho\big(t-s, t(\tilde{h}-h_0) - s(\eta-h_0)\big) \end{matrix}\right| \qquad (434) \\ &\quad\leq T\left|\rho\big(t-s, t(h-h_0) - s(\eta-h_0)\big) - \rho\big(t-s, t(\tilde{h}-h_0) - s(\eta-h_0)\big)\right| \qquad (435) \end{aligned}$$

Therefore by applications of Theorem 7.3 to (**??**) and an application of (**??**) to Theorem 7.4 we have

$$\left| \left(\int_{t_0+\epsilon}^{t} \int_{\mathbb{R}} |K_{h_0}(s,\eta;t,h) - K_{h_0}(s,\eta;\widetilde{t},h)|^{\frac{2M}{M-2}} \, d\eta \, ds \right)^{\frac{M-2}{2M}} \right| \leq C \left| |t-\tilde{t}|^{\frac{M}{2}-1} + |t-\tilde{t}|^{\frac{M}{4}-2} \right| \tag{436}$$

where C is a constant depending on t, T, ϵ and M, since such that $|t-\tilde{t}| \leq 1$, the result follows. ☐

NOTE: needs to be changed Tightness follows from an application of the (E.1) inequality along with the estimates in Equation 436 and .

A Local Limit Theorems

A.1 Local Limit Theorem for $P_{h_n}^{\mathrm{TILT}}$

NOTE: Read One last time

Theorem A.1. *Under the scaling $h_n = hn^{-1/2}$, we have as $n \to \infty$:*

$$P_{h_n}^{\mathrm{TILT}}(S_n = x) = \frac{2}{\sqrt{2\pi n}} \exp\left(-\frac{2\left(x - \frac{h\sqrt{n}}{2}\right)^2}{n} \right) + O\left(n^{-1+o(1)}\right)$$

where $S_n = X_1 + \cdots + X_n$ and for $1 \leq i \leq n$ the X_i are all i.i.d. Bernoulli distributed random variables under the probability measure P.

Remark 2. *It is tempting to directly apply the local limit theorem (see [GK54]) under the tilted measure $P_{h_n}^{\mathrm{TILT}}$. However, this is not formally justified, because under $P_{h_n}^{\mathrm{TILT}}$ the distribution of the increments depends on n. Thus, the tilted random walk does not form a sequence of i.i.d. random variables with a fixed distribution, but rather a triangular array, and the classical local limit theorem does not apply directly in this setting.*

Since the tilted measure is defined via an exponential change of measure with respect to the simple random walk, one can alternatively derive the asymptotics from sharp local limit estimates for the simple random walk and transfer them to the tilted measure. On the diffusive scale $x = O(\sqrt{n})$ this yields the same Gaussian limit, and in fact gives a sharper error term of order $O(n^{-3/2})$, see [Pet75]. However, this argument does not immediately give uniform control of the error on other scales, where the exponential tilt may significantly distort the probabilities.

Proof. We can calculate the expectation and variance of a Bernoulli random variable X under the measure P_h^{TILT}. This is given by

$$\mu_n := \tanh(h_n), \tag{437}$$

$$\sigma_n^2 := \operatorname{sech}^2(h_n). \tag{438}$$

Note that as $n \to \infty$ we have $\tanh(h_n) \to 0$ and $\operatorname{sech}^2(h_n) \to 1$. The third moment can also be explicitly calculated and is given by $P_{h_n}^{\mathrm{TILT}}\left[X^3\right] = \tanh(h_n)$. We can bound the first, second, and third moments of the SRW under the measure P_h^{TILT} by a universal constant which is independent of n. The X_i for the SRW take values in $\{-1, 1\}$, and this leads to the usual parity issues in the local CLT. So instead, we will work with the increments $X_i \in \{0, 1\}$ by making the change of variable $\tilde{X}_i = X_i/2 + 1/2$. Then, we have

$$P_h^{\mathrm{TILT}}(S_n = x) = P_h^{\mathrm{TILT}}\left(\tilde{S}_n = \frac{x+n}{2}\right) \tag{439}$$

where $\tilde{S}_n$ is the random walk with increments $\tilde{X}_i$. Similarly,

$$\tilde{\mu}_n = \frac{\tanh(h_n)}{2} + \frac{1}{2}, \qquad \tilde{\sigma}_n^2 = \frac{\operatorname{sech}^2(h_n)}{4}, \qquad \tilde{x} = \frac{x+n}{2}. \tag{440}$$

We define the limiting variances by

$$\sigma^2 := \lim_{n\to\infty} \sigma_n^2, \qquad \tilde{\sigma}^2 := \lim_{n\to\infty} \tilde{\sigma}_n^2.$$

By Fourier inversion, for integers $\tilde{x}$,

$$\mathbf{1}_{\{\widetilde{S}_n = (x+n)/2\}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it\widetilde{S}_n} e^{-it(x+n)/2}\, dt.$$

Taking expectations with respect to the measure $P_{h_n}^{\mathrm{TILT}}$ and applying Fubini's theorem yields

$$P_{h_n}^{\mathrm{TILT}}\left(\widetilde{S}_n = \tilde{x}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\phi}_{\widetilde{S}_n}^{\mathrm{TILT}}(t) e^{-it(x+n)/2}\, dt,$$

where $\hat{\phi}_{\widetilde{S}_n}^{\mathrm{TILT}}(t)$ is the characteristic function with respect to the measure $P_{h_n}^{\mathrm{TILT}}$. We have the relationship

$$\hat{\phi}_{\widetilde{S}_n}^{\mathrm{TILT}}(t) = \left(\hat{\phi}_{\widetilde{X}-\tilde{\mu}_n}^{\mathrm{TILT}}(t)\right)^n e^{i\tilde{\mu}_n nt}. \tag{441}$$

Thus, we can express $P_{h_n}^{\mathrm{TILT}}\left(\widetilde{S}_n = \tilde{x}\right)$ as

$$P_{h_n}^{\mathrm{TILT}}\left(\widetilde{S}_n = \tilde{x}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\mu_n nt} \left(\hat{\phi}_{\widetilde{X}-\tilde{\mu}_n}^{\mathrm{TILT}}(t)\right)^n e^{-itx/2}\, dt.$$

where we have used $\tilde{\mu}_n = \mu_n/2+1/2$. After making a change of variable $t = \frac{u}{\sqrt{n}}$, we find

$$\sqrt{n}\, P_{h_n}^{\mathrm{TILT}}\left(\widetilde{S}_n = \tilde{x}\right) = \frac{1}{2\pi}\int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} e^{i\sqrt{n}\mu_n u}\left(\hat{\phi}_{\widetilde{X}-\tilde{\mu}_n}^{\mathrm{TILT}}\left(\frac{u}{\sqrt{n}}\right)\right)^n e^{-ixu/(2\sqrt{n})}\,du. \tag{442}$$

Let us heuristically analyse the first two terms in the integrand of (442):

$$e^{i\sqrt{n}\mu_n u/2} \tag{443}$$

$$\left(\hat{\phi}_{\widetilde{X}-\tilde{\mu}_n}^{\mathrm{TILT}}\left(\frac{u}{\sqrt{n}}\right)\right)^n \tag{444}$$

For (444), we note that it is the characteristic function of a sum of iid mean 0 random variables with variance $\tilde{\sigma}_n^2 \to \tilde{\sigma}^2 = 1/4$ (see (437) and just below (439)). By the Lindeberg–Feller central limit theorem [Dur19, Theorem 3.14.10], we have

$$\left(\hat{\phi}_{\widetilde{X}-\tilde{\mu}_n}^{\mathrm{TILT}}\left(\frac{u}{\sqrt{n}}\right)\right)^n \to e^{-\tilde{\sigma}^2 u^2/2} \qquad \text{as } n \to \infty.$$

The Taylor series expansion $\tanh(x) = x - O(x^3)$ in (437) implies that for (443) the term

$$e^{i\sqrt{n}\mu_n u} \to e^{ihu/2} \qquad \text{as } n \to \infty.$$

Therefore, one would expect that (but this does not prove that)

$$\sqrt{n}\, P_{h_n}^{\mathrm{TILT}}\left(\widetilde{S}_n = \tilde{x}\right) \to \frac{1}{2\pi}\int_{\mathbb{R}} e^{ihu} e^{-\tilde{\sigma}^2 u^2/2} e^{-ixu/(2\sqrt{n})}\,du. \tag{445}$$

Note that we can explicitly calculate the integral on the right-hand side recalling that $\tilde{\sigma^2} = 1/4$ Completing the square, we get

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{ihu/2} e^{-u^2/8} e^{-ixu/\sqrt{n}}\,du = \sqrt{\frac{2}{\pi}}\exp\left(-\frac{2\left(x - \frac{h\sqrt{n}}{2}\right)^2}{n}\right). \tag{446}$$

We return to (442). What remains to be shown is that we can get an appropriate bound on the difference of the following integrals, that is

$$\left|\frac{1}{2\pi}\int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} e^{i\sqrt{n}\mu_n u}\left(\hat{\phi}_{X-\mu_n}^{\mathrm{TILT}}\left(\frac{u}{\sqrt{n}}\right)\right)^n e^{-ixu/\sqrt{n}}\,du - \frac{1}{2\pi}\int_{\mathbb{R}} e^{ihu} e^{-\tilde{\sigma}^2 u^2/2} e^{-ixu/(2\sqrt{n})}\,du\right|$$

$$\leq \int_{-\sqrt{n}\pi}^{\sqrt{n}\pi}\left|\left(e^{i\sqrt{n}\mu_n u}\left(\hat{\phi}_{X-\mu_n}^{\mathrm{TILT}}\left(\frac{u}{\sqrt{n}}\right)\right)^n - e^{ihu}e^{-\tilde{\sigma}^2 u^2/2}\right)\right| du \tag{447}$$

$$+\left|\int_{|u|\geq\sqrt{n}\pi} e^{ihu} e^{-\tilde{\sigma}^2 u^2/2} e^{-ixu/(2\sqrt{n})}\,du\right|. \tag{448}$$

by the triangle inequality. Throughout the rest of the proof, we let C_k be constants indexed by k which will change line by line. We bound equation (448) first. Using the inequality $\Phi(t) \leq \frac{1}{t\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ for $t > 0$, where $\Phi(t)$ is the cdf of the standard Gaussian, we obtain

$$\left| \int_{|u| \geq \sqrt{n}\pi} e^{ihu} e^{-\frac{\tilde{\sigma}^2 u^2}{2}} e^{-\frac{ixu}{2\sqrt{n}}} \, du \right| \leq \int_{|u| \geq \sqrt{n}\pi} e^{-\frac{\tilde{\sigma}^2 u^2}{2}} \, du \leq C_1 e^{-nC_2}$$

For equation (447), adding and subtracting $e^{-\frac{\tilde{\sigma}^2 u^2}{2}} e^{i\sqrt{n}\mu_n u}$ and using the triangle inequality gives us

$$\int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} \left| \left(e^{i\sqrt{n}\mu_n/2u} \left(\hat{\phi}^{\text{TILT}}_{\tilde{X}-\tilde{\mu_n}} (\frac{u}{\sqrt{n}}) \right)^n - e^{ihu} e^{-\frac{\tilde{\sigma}^2 u^2}{2}} \right) - e^{-\frac{\tilde{\sigma}^2 u^2}{2}} e^{i\sqrt{n}\mu_n u} + e^{-\frac{\tilde{\sigma}^2 u^2}{2}} e^{i\sqrt{n}\mu_n u} \right| du$$

$$\leq \int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} \left| \left(\hat{\phi}^{\text{TILT}}_{\tilde{X}-\tilde{\mu_n}} \left(\frac{u}{\sqrt{n}} \right)^n - e^{-\frac{\tilde{\sigma}^2 u^2}{2}} \right) \right| du \tag{449}$$

$$+ \qquad \int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} \left| e^{-\frac{\tilde{\sigma}^2 u^2}{2}} \left(e^{ihu} - e^{i\sqrt{n}\mu_n u} \right) \right| du \tag{450}$$

For (450), we note that

$$\begin{aligned} \int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} \left| e^{-\frac{\tilde{\sigma}^2 u^2}{2}} \left(e^{ihu} - e^{i\sqrt{n}\mu_n u} \right) \right| du &= \int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} \left| e^{-\frac{\tilde{\sigma}^2 u^2}{2}} \right| \left| e^{ihu} - e^{i\sqrt{n}\mu_n u} \right| du \\ &\leq \int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} |u| e^{-\frac{\tilde{\sigma}^2 u^2}{2}} \left| h - \sqrt{n}\mu_n \right| du \\ &\leq \frac{C_3}{n}, \end{aligned} \tag{451}$$

where we have used the inequality

$$|e^{ihu} - e^{i\sqrt{n}\mu_n u}| = \left| \int_{\sqrt{n}\mu_n u}^{hu} i e^{iy} \, dy \right| \leq |u| \left| h - \sqrt{n}\mu_n \right|$$

and applied Taylor's theorem to $\tanh(x) = x - O(x^3)$ to obtain

$$\sqrt{n}\mu_n = h + O\left(\frac{1}{n}\right).$$

For equation (449), we consider an $\epsilon \ll 1$ and break up the region of integration into two parts

$$R_1 := \{u : |u| < \epsilon\sqrt{n}\}, \tag{452}$$

$$R_2 := \{u : \epsilon\sqrt{n} \leq |u| \leq \pi\sqrt{n}\}, \tag{453}$$

and apply two different estimates in each region. The region (453) is compact. We claim that since the characteristic function is continuous, there exists a constant $C_4 < 1$, such that

$$\left|\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)^n\right| < C_4^n \tag{454}$$

in (453). Recall the definition of the characteristic function

$$\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}(t) = E^{\mathrm{TILT}}_{h_n}\left[e^{it(\tilde{X}-\tilde{\mu}_n)}\right]$$

$$\hat{\phi}^{\mathrm{SRW}}_{\tilde{X}-\tilde{\mu}_n}(t) = E^{SRW}\left[e^{it(\tilde{X}-\tilde{\mu}_n)}\right].$$

We consider the difference between the characteristic function under the original measure P^{SRW} and the characteristic function under the tilted measure $P^{\mathrm{TILT}}_{h_n}$. Note that for this computation, we can temporarily ignore the subtraction of the mean $\tilde{\mu}_n$, since $|e^{iu\tilde{\mu}_n/\sqrt{n}}| = 1$.

$$\begin{aligned}
&\left|\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}}(u/\sqrt{n}) - \phi^{\mathrm{SRW}}_{\tilde{X}}(u/\sqrt{n})\right| \\
&= \left|E^{\mathrm{SRW}}\left[\frac{e^{h_n X}e^{i\tilde{X}u/\sqrt{n}} - e^{i\tilde{X}u/\sqrt{n}}\cosh(h_n)}{\cosh(h_n)}\right]\right| \\
&\leq E^{\mathrm{SRW}}\left[\left|\frac{e^{h_n X}e^{i\tilde{X}u/\sqrt{n}} - e^{i\tilde{X}u/\sqrt{n}}\cosh(h_n)}{\cosh(h_n)}\right|\right] \\
&= E^{\mathrm{SRW}}\left[\left|\frac{e^{h_n X} - \cosh(h_n)}{\cosh(h_n)}\right|\right].
\end{aligned} \tag{455}$$

Since we have that our random variable X is bounded, a Taylor series expansion shows that $e^{h_n X}/\cosh(h_n) - 1 = O(h_n)$, as we have that

$$(455) \leq \frac{C_5}{\sqrt{n}}$$

for some large enough n, where C_5 is a constant which depends on h. Since the original characteristic function $|\phi^{\mathrm{SRW}}_{\tilde{X}}(u/\sqrt{n})|$ is bounded by a constant C_7, which is less than or equal to one, an application of the triangle inequality shows that

$$|\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}(u/\sqrt{n})| \leq \frac{C_6}{\sqrt{n}} + C_7 \leq 1$$

for large enough n (we have just used the inequality $|A| \leq |A-B| + |B|$). We now show that in region (453) the constant c is not equal to 1. We can chose n large enough such that

$$C_4 = C_5/\sqrt{n} + C_6 < 1.$$

Now we show that $C_7 \neq 1$. Suppose for a contradiction that $\left|\hat{\phi}^{\mathrm{SRW}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)\right| = 1$ for some $u \in R_2$. Then $e^{i(u/\sqrt{n})\tilde{X}}$ is constant almost surely. Since $\tilde{X} \in \{0, 1\}$, this implies $e^{iu/\sqrt{n}} = 1$, and hence $\frac{u}{\sqrt{n}} \in 2\pi\mathbb{Z}$. But on R_2 we have $\epsilon \leq |u|/\sqrt{n} \leq \pi$, which is a contradiction. Therefore $\left|\hat{\phi}^{\mathrm{SRW}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)\right| < C_4$ for all $u \in R_2$. Hence, by an application of the triangle inequality, for equation (449) in the region (453) we have

$$\int_{-\sqrt{n}\pi}^{\sqrt{n}\pi} \left|\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)^n - e^{-\frac{\tilde{\sigma}^2 u^2}{2}}\right| \mathbf{1}_{R_2}\, du \leq e^{-C_7 n}. \tag{456}$$

In the region (452) for equation (449), more careful analysis is required. We consider splitting region (452) into two further regions. These are given by

$$R_{1a} := \{u : |u| < \epsilon n^{\delta}\}, \tag{457}$$
$$R_{1b} := \{u : -\epsilon\sqrt{n} < u < -\epsilon n^{\delta} \text{ or } \epsilon n^{\delta} \leq u < \epsilon\sqrt{n}\}, \tag{458}$$

where $\delta < 1/6$ and will be optimised later. Let us consider equation (449) in the region (457):

$$\begin{aligned}
&\int_{-\epsilon n^{\delta}}^{\epsilon n^{\delta}} \left|\left(\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)\right)^n - e^{-\frac{\tilde{\sigma}^2 u^2}{2}}\right| du \\
&= \int_{-\epsilon n^{\delta}}^{\epsilon n^{\delta}} \left|e^{n\log\left(\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)\right)} - e^{-\frac{\tilde{\sigma}^2 u^2}{2}}\right| du \\
&= \int_{-\epsilon n^{\delta}}^{\epsilon n^{\delta}} \left|e^{-\frac{\tilde{\sigma}^2 u^2}{2}}\right| \left|1 - e^{n\log\left(\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)\right) + \frac{\tilde{\sigma}^2 u^2}{2}}\right| du.
\end{aligned}$$

First we apply a Taylor series expansion of our characteristic function up to the third moment,

$$\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right) = 1 - \frac{\tilde{\sigma}_n^2 u^2}{2n} + O\left(n^{-\left(\frac{3}{2}-3\delta\right)}\right).$$

After another application of Taylor's theorem (recall $\log(1+x) = x + O(x^2)$),

$$\log\left(\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)\right) = -\frac{\tilde{\sigma}_n^2 u^2}{2n} + O\left(n^{-\left(\frac{3}{2}-3\delta\right)}\right).$$

Hence, multiplying by n gives

$$n\log\left(\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)\right) = -\frac{\tilde{\sigma}_n^2 u^2}{2} + O\left(n^{-\left(\frac{1}{2}-3\delta\right)}\right). \tag{459}$$

Since

$$\tilde{\sigma}_n^2 = \frac{\mathrm{sech}^2(h_n)}{4} \quad \text{and} \quad h_n = O(n^{-1/2}),$$

a Taylor expansion of sech^2 around 0 yields

$$\tilde{\sigma}_n^2 = \tilde{\sigma^2} + O(n^{-1}), \qquad \text{where } \tilde{\sigma}^2 = 1$$

Hence, on the region $|u| \leq \epsilon n^{\delta}$,

$$\left| e^{-\tilde{\sigma}_n^2 u^2/2} - e^{-\sigma^2 u^2/2} \right| \leq |u|^2 |\tilde{\sigma}_n^2 - \sigma^2| e^{-C_6 u^2} \leq C_8 n^{2\delta-1} e^{-C_9 u^2}, \tag{460}$$

which tends to 0 uniformly since $\delta < 1/2$. Combining (460) and (459) we obtain

$$n \log\Big(\hat{\phi}^{\text{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)\Big) + \frac{\tilde{\sigma_n}^2 u^2}{2} = O\Big(n^{-\left(\frac{1}{2}-3\delta\right)}\Big).$$

Hence our integral becomes

$$\begin{aligned}
&\int_{-\epsilon n^{\delta}}^{\epsilon n^{\delta}} \left| e^{-\frac{u^2\sigma^2}{2}} \right| \left| 1 - e^{O\left(n^{-\left(\frac{1}{2}-3\delta\right)}\right)} \right| du \\
&\leq \int_{-\epsilon n^{\delta}}^{\epsilon n^{\delta}} \left| e^{-\frac{u^2\sigma^2}{2}} \right| \left| \frac{C_7}{n^{\left(\frac{1}{2}-3\delta\right)}} \right| du \leq \frac{C_8}{n^{\left(\frac{1}{2}-3\delta\right)}}.
\end{aligned} \tag{461}$$

Where we have used the Taylor expansion of the exponential near 0 to obtain

$$|1 - e^z| \leq C_9 |z|$$

for sufficiently small z. Choosing $\delta < 1/6$, there is sufficient decay of order $n^{3\delta - 1/2}$. For region (458) one can use the standard bound on the characteristic function found in [Dur19, Theorem 3.3.20]. This is given by

$$\hat{\phi}^{\text{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right) = 1 - \frac{\tilde{\sigma}_n^2 u^2}{2n} + O\left(E^{\text{TILT}}\left[\left| \left(\frac{u}{\sqrt{n}}(\tilde{X} - \tilde{\mu}_n)\right)^3 \right| \wedge \left| \left(\frac{u}{\sqrt{n}}(\tilde{X} - \tilde{\mu}_n)\right)^2 \right| \right] \right). \tag{462}$$

We claim that for u in the region (458) the minimizing term in (462) is given by the cubic term as long as $|\tilde{X}\| \leq 1/\epsilon$. Consequently, we have that

$$\begin{aligned}
&E_h^{\text{TILT}}\left[\left| \left(\frac{u}{\sqrt{n}}\tilde{X}\right)^3 \right| \wedge \left| \left(\frac{u}{\sqrt{n}}\tilde{X}\right)^2 \right| \right] \\
&\leq E_{h_n}^{\text{TILT}}\left[(\epsilon\tilde{X})^3 \mathbf{1}_{\{\tilde{X} < 1/\epsilon\}} \right] + E_{h_n}^{\text{TILT}}\left[(\epsilon\tilde{X}^2) \mathbf{1}_{\{\tilde{X} > 1/\epsilon\}} \right] \\
&\leq \epsilon^3 E_{h_n}^{\text{TILT}}\left[\tilde{X}^3 \right] + E_{h_n}^{\text{TILT}}\left[(\epsilon\tilde{X})^3 \right]^{2/3} E_{h_n}^{\text{TILT}}\left(\tilde{X} > 1/\epsilon \right)^{1/3} \\
&\leq \epsilon^3.
\end{aligned}$$

Where we have used Hölder's inequality with exponents $p = 3/2$ and $q = 3$ in the penultimate inequality, and the last inequality follows from an application of Markov's inequality along with the uniform bounds for the third moment. Hence, we get the inequality

$$\left|\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)\right|^n \le \left(1-\frac{u^2\tilde{\sigma}_n^2}{2n}+\frac{u^2\tilde{\sigma}_n^2}{4n}\right)^n = \left(1-\frac{u^2\tilde{\sigma}_n^2}{4n}\right)^n \le C_{10}e^{-C_{11}u^2}.$$

using the inequality $1-x\le e^{-x}$ and the uniform bounds from below of the variance $\tilde{\sigma^2}$. Consequently in region R_{1b} (see (458)), we have

$$\int_{R_{1b}}\left|\hat{\phi}^{\mathrm{TILT}}_{\tilde{X}-\tilde{\mu}_n}\left(\frac{u}{\sqrt{n}}\right)^n - e^{-\frac{\sigma^2u^2}{2}}\right| du \le C_{12}\sqrt{n}e^{-C_{13}n^{2\delta}}. \tag{463}$$

Combining the errors in regions R_{1a}, R_{1b} and R_2 in equations (461), (463), and (454) respectively, we see that the dominant error term is $O(n^{-(1/2-3\delta)})$. Dividing by $\sqrt{n}$ in (442) provides an error of $O(n^{-1+3\delta})$.

□

A.2 Discrete bounds $P^{\mathrm{TILT}}_{h_n}$

Lemma A.2. *The tilted heat kernel is defined as*

$$p_n^{\mathrm{TILT}}(t,x) := \sqrt{n}\,P^{\mathrm{TILT}}\left(S_{\lfloor nt\rfloor} = \lfloor x\sqrt{n}\rfloor\right). \tag{464}$$

There exist constants $C,c>0$ such that

$$p_n^{\mathrm{TILT}}(t,x) \le \frac{C}{\sqrt{t}}\exp\left(-c\,\frac{\left(\lfloor x\sqrt{n}\rfloor - (h\sqrt{n})/2\rfloor\right)^2}{nt}\right). \tag{465}$$

for all $t>0$ and all $x\in\mathbb{R}$.

Remark 3. ***NOTE:*** *I probably need a multidimensional version The bound (465) indicates that p_n^{TILT} is controlled by a shifted Gaussian heat kernel. For $h_n = h/\sqrt{n}$, the diffusive scaling of the right-hand side of (465) behaves as*

$$\frac{C}{\sqrt{t}}\exp\left(-c\,\frac{(x-ht)^2}{t}\right),$$

ignoring discretisation errors. Since the exponential is bounded by 1, we have $p_n^{\mathrm{TILT}}(t,x)\le C/\sqrt{t}$. Further, for $t\in[\varepsilon,T]$, the kernel is uniformly bounded: $p_n^{\mathrm{TILT}}(t,x) = O(1)$.

Proof. Recall the defintion of μ_n and σ_n from (437). We split our analysis into a diffusive region given by

$$|x-n\mu_n| \le C\sqrt{n} \tag{466}$$

and its complement

$$|x-n\mu_n| \ge C\sqrt{n}. \tag{467}$$

By the local limit theorem Theorem A.1, which we recall has an error term $\mathfrak{E}_n$ that is uniform in x, we have

$$P_{h_n}^{\mathrm{TILT}}(S_n = x) = \frac{2}{\tilde{\sigma}_n\sqrt{2\pi n}}\exp\left(-\frac{(x-n\tilde{\mu}_n)^2}{2\tilde{\sigma}_n^2 n}\right) + R_n \tag{468}$$

where $R_n \to 0$ as $n \to \infty$. In the diffusive regime (466), we have that

$$\exp\left(-\frac{(x-n\tilde{\mu}_n)^2}{2\tilde{\sigma}_n^2 n}\right) \geq \exp(-c^2), \tag{469}$$

for some constant $c > 0$ depending only on C since we have uniform bounds on $\tilde{\sigma}_n$, see . Therefore, plugging (469) into (468) we obtain that

$$P_{h_n}^{\mathrm{TILT}}(S_n = x) \leq \left(1 + \frac{R_n}{e^{-c^2}}\right)\frac{2}{\sigma_n\sqrt{2\pi n}}\exp\left(-\frac{(x-n\mu_n)^2}{2\tilde{\sigma}_n^2 n}\right). \tag{470}$$

Choosing n large enough so that $\frac{R_n}{e^{-c^2}} \leq 1$ and recalling that $\mu_n = x - O(x^3)$ yields the result in the region $|x - n\mu_n| \leq C\sqrt{n}$. For the region (467) we note that by Azuma's inequality, if $|x - n\mu_n| > C\sqrt{n}$, then under the measure $P_{(}^{\mathrm{TILT}}h_n)$, $M_k := S_k - k\mu_n$ is a martingale with bounded increments

$$|M_k - M_{k-1}| \leq 2. \tag{471}$$

Hence Azuma's inequality yields

$$P_{h_n}^{\mathrm{TILT}}\big(|S_n - n\mu_n| \geq r\big) \leq 2\exp\left(-\frac{r^2}{8n}\right).$$

In particular,

$$P_{h_n}^{\mathrm{TILT}}(S_n = x) \leq 2\exp\left(-c\,\frac{(x-n\mu_n)^2}{n}\right).$$

Since $|x - n\mu_n| > C\sqrt{n}$, after adjusting the constants this may be rewritten as

$$P_{h_n}^{\mathrm{TILT}}(S_n = x) \leq \frac{C}{\sqrt{n}}\exp\left(-c'\,\frac{(x-n\mu_n)^2}{n}\right).$$

Combining the two regions proves the claim by replacing x with $x\sqrt{n}$ and n with nt. $\square$

Corollary A.3. *Let C, c, k be positive constants. Let $\delta > 0$. Then for all $t > 0$ and $x \in \mathbb{R}$,*

$$\left|p_n^{\mathrm{TILT}}(t,x) - p_n^{\mathrm{TILT}}(t,x+\delta)\right| \le C\left[\exp\left(-c\frac{(x-ht)^2}{t}\right) + \exp\left(-k\frac{(x+\delta-ht)^2}{t}\right)\right], \tag{472}$$

$$\left|p_n^{\mathrm{TILT}}(t,x) - p_n^{\mathrm{TILT}}(t+\delta,x)\right| \le C\left[\exp\left(-c\frac{(x-ht)^2}{t}\right) + \exp\left(-k\frac{(x-h(t+\delta))^2}{t+\delta}\right)\right]. \tag{473}$$

Proof. Both proofs follow from an application of the triangle and the proof of (A.2). The bound on the LHS of (472) is justified by (3). □

A.3 Local Limit Theorem For The Velocity Measure

Lemma A.4 (Local limit theorem for the transformed walk)**.** *Let $H^{(N)}$ be the discrete process defined by*

$$H_k^{(N)} = h_0 + \frac{S_k}{\sqrt{N}\,t_k}, \qquad t_k = \frac{k}{N}. \tag{474}$$

Fix $0 < s < t$, and let $i = \lfloor Ns \rfloor$, $j = \lfloor Nt \rfloor$ Then we have the following convergence ***NOTE:*** *what exactly is the convergence here*

$$\frac{\sqrt{N}\,t}{2} P\left(H_j^{(N)} = h_N \,\middle|\, H_i^{(N)} = \eta_N\right) \longrightarrow K_{h_0}(s,\eta;t,h), \tag{475}$$

where

$$K_{h_0}(s,\eta;t,h) = \frac{t}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{t^2}{2(t-s)}\left(h - \frac{s}{t}\eta - \left(1-\frac{s}{t}\right)h_0\right)^2\right\}. \tag{476}$$

Proof. After rearrangement, the event $H_i^{(N)} = \eta$ can be written in the form:

$$S_i = \sqrt{N} t_i(\eta_N - h_0).$$

Similarly, the event $H_j^{(N)} = h_N$ can be expressed as:

$$S_j = \sqrt{N} t_j(h_N - h_0).$$

Using the Markov property of the underlying simple random walk (SRW), we have:

$$P\left(H_j^{(N)} = h \,\middle|\, H_i^{(N)} = \eta\right) = P\left(S_{j-i} = \sqrt{N} t_j(h_N - h_0) - \sqrt{N} t_i(\eta_N - h_0)\right). \tag{477}$$

By definition, we have $j - i = \lfloor Nt \rfloor - \lfloor Ns \rfloor$. We note that as $N \to \infty$:

$$\frac{j-i}{N} \to t - s. \quad \text{[Check indices and sign here]} \tag{478}$$

By the local central limit theorem for the simple symmetric random walk (see [Pet75]), it follows that:

$$\frac{\sqrt{N}t}{2} P\left(H_j^{(N)} = h_N \,\middle|\, H_i^{(N)} = \eta_N\right) = \frac{t}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(t(h-h_0) - s(\eta - h_0))^2}{2(t-s)}\right\} + O(N^{-1}). \tag{479}$$

NOTE: fix overfill $\square$

B The results of Alberts, Khanin and Quastel in [AKQ14b].

Proposition B.1 (Theorem 2.1 in [AKQ14b]). *Under the assumption that the random environment ω satisfies $\lambda(\beta) < \infty$, for small β we have that*

$$e^{-n\lambda(\beta n^{-1/4})} \frac{\sqrt{n}}{2} Z_n(x\sqrt{n}) \xrightarrow[n\to\infty]{(d)} \frac{e^{A_{\sqrt{2}\beta}(x)} e^{-x^2/2}}{\sqrt{2\pi}}. \tag{480}$$

where the convergence in distribution is given by the topology of the supremum norm on the space of bounded continuous functions. The process $x \to A_{\sqrt{2}\beta}(x)$ is a one parameter family of stationary processes whose one point marginals have explicit formulas given in [ACQ11]. Further, the process $e^{A_{\sqrt{2}\beta}(x)}$ can be written in the form of a wick product *(see [Jan97] and therein for more references)*

$$e^{A_{\sqrt{2}\beta}(x)} := \mathbb{E}_{\mathrm{BB}}\left[: \exp : \left\{ \sqrt{2}\beta \int_0^1 \xi(s, X_s + xs)\, ds \right\} \right], \tag{481}$$

where $\xi(x,t)$ is space-time white noise and the expectation $\mathbb{E}_{BB}$ is given over Brownian bridges X_t that start and end at zero, in a unit time interval.

Proposition B.2. *The process*

$$(x,t) \mapsto z_n(x,t) := \sqrt{n}\left(nt, x\sqrt{n}, \beta n^{-1/4}\right) \tag{482}$$

is continuous in space and time, and is tight as a random element of $C([\epsilon, T] \times \mathbb{R})$ for any $0 < \epsilon < T < \infty$. Furthermore, $z_n(x,t)$ is α-Hölder continuous for $0 < \alpha < 1/4$ in time and β-Hölder continuous for $0 < \beta < 1/2$ in space.

C Some Theory on the Parabolic Anderson Model

Theorem C.1 (Result (5) of [Flo14][1]). *Let Z be a continuous modification of the solution to the Parabolic Anderson Model*

$$\partial_t Z(t,x) = \frac{1}{2}\Delta Z(t,x) + \beta Z(t,x)\xi(t,x), \qquad t>0,\ x\in\mathbb{R}, Z(0,x) \quad = \delta_0(x),$$

where ξ denotes space-time white noise. Then

$$\mathbb{P}\big(Z(t,x)=0 \text{ for some } t>0,\ x\in\mathbb{R}\big) = 0. \tag{483}$$

Next we have a uniqueness theorem, which is due to [BC95].

Theorem C.2 (Theorem 2.2 in [BC95]). *[**?**] Let $\{\psi_t(x) : t>0,\ x\in\mathbb{R}\}$ be a continuous $\mathcal{F}_t$-adapted process[2] such that for every $T>0$,*

$$\sup_{t\in(0,T]}\sup_{x\in\mathbb{R}}\int_0^t\int_0^s\int_{\mathbb{R}}\int_{\mathbb{R}}\rho(t-s,x-y)^2\rho(s-s',y-y')^2\,\mathbb{E}\big[\psi_{s'}(y')^2\big]\ dy'\,dy\,ds'\,ds<\infty. \tag{484}$$

Then $\psi_t(x)$ is the mild solution of the Parabolic Anderson Model

$$\partial_t Z(t,x) = \frac{1}{2}\Delta Z(t,x) + \beta Z(t,x)\,\xi(t,x), \qquad t>0,\ x\in\mathbb{R}, \tag{485}$$

$$Z(0,x) = \delta_0(x). \tag{486}$$

Proof. We first recall the classical occupation times formula: if Z is a continuous semimartingale with local time L_s^z, then for any bounded Borel function $f:\mathbb{R}\to\mathbb{R}$ and $t_0\le u<v$,

$$\int_u^v f(Z_s)\,d\langle Z\rangle_s = \int_{\mathbb{R}} f(z)\left(L_v^z - L_u^z\right)dz \qquad \text{a.s.} \tag{487}$$

D Volterra Type Inequalities

Lemma D.1. *For every even integer $M>6$, there exists a constant C_M such that*

$$\mathbb{E}\big[\,|\overline{z}_n(s,y)|^M\big] \le C_M\big(p_n^{\mathrm{TILT}}(s,y)\big)^{2M}. \tag{488}$$

[1]This was first proved by [Mue91] and later extended by [BC95] in the case of Dirac initial data.

[2]Here $\mathcal{F}_t$ denotes the filtration generated by the space–time white noise (equivalently, the associated cylinder σ-algebra).

Proof. Recall (390). Applying the inequality $|a+b|^{2M} \le C_M\big(|a|^{2M} + |b|^{2M}\big)$ together with the Burkholder–Davis–Gundy inequality, we obtain

$$\mathbb{E}\big[\,|\overline{z}_n(s,y)|^{2M}\big] \le C_M\big(p_n^{\mathrm{TILT}}(s,y)\big)^{2M} + C_M\mathbb{E}\left[\left(\beta\int_\epsilon^t\int_{\mathbb{R}}\big(p_n^{\mathrm{TILT}}(t-r,y-z)\big)^2\overline{z}_n(r,z)^2\,dz\,dr\right)^M\right]. \tag{489}$$

Notice that raising to the m-th power produces a product over $1 \le i \le m$, in which each term is symmetric. Therefore, we may rewrite the second term as an integral over the simplex. In particular, we can rewrite (489) as

$$C_M\int_{\Delta_M}\int_{\mathbb{R}^M}\prod_{i=1}^M\big(p_n^{\mathrm{TILT}}(t-r_i,y-z_i)\big)^2\,\mathbb{E}\left[\prod_{i=1}^M\overline{z}_n(r_i,z_i)^2\right]dz_1\cdots dz_M\,dr_1\cdots dr_M \tag{490}$$

where $\Delta_M := \{(r_1,\ldots,r_M)\in[\epsilon,t]^M : \epsilon \le r_1 \le \cdots \le r_M \le t\}$. An application of Hölder's inequality gives

$$\mathbb{E}\left[\prod_{i=1}^M\overline{z}_n(r_i,z_i)^2\right] \le \prod_{i=1}^M\mathbb{E}\big[\,|\overline{z}_n(r_i,z_i)|^{2M}\big]^{1/M}. \tag{491}$$

Using (491) in (490), we obtain

$$C_M\int_{\Delta_M}\int_{\mathbb{R}^M}\prod_{i=1}^M\big(p_n^{\mathrm{TILT}}(t-r_i,y-z_i)\big)^2 \times\prod_{i=1}^M\mathbb{E}\big[\,|\overline{z}_n(r_i,z_i)|^{2M}\big]^{1/M}\,dz_1\cdots dz_M\,dr_1\cdots dr_M. \tag{492}$$

Now define the ratio

$$g(t,x) = \frac{\mathbb{E}\big[\,|\overline{z}_n(t,x)|^{2M}\big]}{\big(p_n^{\mathrm{TILT}}(t,x)\big)^{2M}}. \tag{493}$$

Dividing the moment inequality by $\big(p_n^{\mathrm{TILT}}(t,x)\big)^{2M}$ and using (492), we obtain

$$g(t,x) \le C_M\left(1+\int_{\Delta_M}\int_{\mathbb{R}^M}\frac{\prod_{i=1}^M\big(p_n^{\mathrm{TILT}}(t-r_i,y-z_i)\big)^2}{\big(p_n^{\mathrm{TILT}}(t,x)\big)^{2M}} \times\prod_{i=1}^M\big(p_n^{\mathrm{TILT}}(r_i,z_i)\big)^2\,g(r_i,z_i)^{1/M}\,dz_1\cdots dz_M\,dr_1\cdots dr_M\right). \tag{494}$$

By the Chapman–Kolmogorov identity, we have

$$\sum_{z\in\mathbb{Z}} p_n^{\mathrm{TILT}}(t-s,x-z)\, p_n^{\mathrm{TILT}}(s,z) = p_n^{\mathrm{TILT}}(t,x). \tag{495}$$

Dividing both sides by $p_n^{\mathrm{TILT}}(t,x)$ yields

$$\sum_{z\in\mathbb{Z}} \frac{p_n^{\mathrm{TILT}}(t-s,x-z)\, p_n^{\mathrm{TILT}}(s,z)}{p_n^{\mathrm{TILT}}(t,x)} = 1. \tag{496}$$

Thus, defining

$$\Pi_n(s,z;t,x) := \frac{p_n^{\mathrm{TILT}}(t-s,x-z)\, p_n^{\mathrm{TILT}}(s,z)}{p_n^{\mathrm{TILT}}(t,x)}, \tag{497}$$

we see that $\Pi_n(s,z;t,x)$ is a probability mass function in z. Consequently, we may rewrite

$$\frac{p_n^{\mathrm{TILT}}(t-r,x-z)^2\, p_n^{\mathrm{TILT}}(r,z)^2}{\left(p_n^{\mathrm{TILT}}(t,x)\right)^2} := \Pi_n(r,z;t,x)^2, \tag{498}$$

which allows us to control the integral by summing against a probability weight. Hence the normalized product of transition kernels may be interpreted as a bridge density. Using this Chapman–Kolmogorov identity and the definition of Π_n, we obtain

$$\begin{aligned} g(t,x) \le C_M \Bigg(1 + \int_{\Delta_M} \int_{\mathbb{R}^M} \prod_{i=1}^{M} \Pi_n(r_i,z_i;t,x)^2 \\ \times \prod_{i=1}^{M} g(r_i,z_i)^{1/M}\, dz_1 \cdots dz_M\, dr_1 \cdots dr_M \Bigg). \end{aligned} \tag{499}$$

$$\Pi_n(r,z;t,x) := \frac{p_n^{\mathrm{TILT}}(t-r,x-z)\, p_n^{\mathrm{TILT}}(r,z)}{p_n^{\mathrm{TILT}}(t,x)}$$

and

$$\Delta_M = \{(r_1,\ldots,r_M) \in [\epsilon,t]^M : \epsilon \le r_1 \le \cdots \le r_M \le t\}.$$

□

Lemma D.2. *Let $q \ge 2$ and let $g(r_i,z_i) > 0$. Then there exists a constant $C > 0$ such that*

$$\prod_{i=1}^{q} g(r_i,z_i)^{1/q} \;\le\; C \sum_{i=1}^{q} \frac{\prod_{j\neq i} p^{\frac{2}{q-1}}(r_j,z_j)}{p^2(r_i,z_i)}\, g(r_i,z_i). \tag{500}$$

Proof. We begin by introducing weights based on the heat kernel. Write

$$\prod_{i=1}^{q} g(r_i, z_i)^{1/q} = \prod_{i=1}^{q} \Big(g(r_i, z_i)\, p^{-2}(r_i, z_i) \Big)^{1/q} \cdot \prod_{i=1}^{q} p^{2/q}(r_i, z_i).$$

Applying the arithmetic–geometric mean inequality,

$$\prod_{i=1}^{q} a_i^{1/q} \le \frac{1}{q} \sum_{i=1}^{q} a_i,$$

with $a_i = g(r_i, z_i)\, p^{-2}(r_i, z_i)$, we obtain

$$\prod_{i=1}^{q} \Big(g(r_i, z_i)\, p^{-2}(r_i, z_i) \Big)^{1/q} \le \frac{1}{q} \sum_{i=1}^{q} g(r_i, z_i)\, p^{-2}(r_i, z_i).$$

Hence,

$$\prod_{i=1}^{q} g(r_i, z_i)^{1/q} \le \frac{1}{q} \sum_{i=1}^{q} g(r_i, z_i)\, p^{-2}(r_i, z_i) \prod_{j=1}^{q} p^{2/q}(r_j, z_j).$$

Now, for each fixed i, we rewrite

$$\prod_{j=1}^{q} p^{2/q}(r_j, z_j) = p^{2/q}(r_i, z_i) \prod_{j \neq i} p^{2/q}(r_j, z_j) \le C \prod_{j \neq i} p^{\frac{2}{q-1}}(r_j, z_j),$$

where we used that $q \ge 2$ and absorbed constants into C. Substituting this into the previous inequality yields

$$\prod_{i=1}^{q} g(r_i, z_i)^{1/q} \le C \sum_{i=1}^{q} \frac{\prod_{j \neq i} p^{\frac{2}{q-1}}(r_j, z_j)}{p^2(r_i, z_i)}\, g(r_i, z_i),$$

which proves (500). $\square$

Applying Lemma D.2 together with (499), we obtain

$$g(t,x) \le C_M \Bigg(1 + \int_{\Delta_M} \int_{\mathbb{R}^M} \prod_{i=1}^{M} \Pi_n(r_i, z_i; t, x)^2 \sum_{i=1}^{M} \frac{\prod_{j \neq i} p(r_j, z_j)^{\frac{2}{q-1}}}{p(r_i, z_i)^2}\, g(r_i, z_i)\, dz_1 \cdots dz_M\, dr_1 \cdots dr_M \Bigg). \tag{501}$$

For each fixed $i \in \{1, \ldots, M\}$, set $r_i = s$ and $z_i = y$. Using

$$\Pi_n(r_i, z_i; t, x)^2\, \frac{1}{p(r_i, z_i)^2} = \frac{p(t - r_i, x - z_i)^2}{p(t, x)^2}, \tag{502}$$

the i-th contribution may be written as

$$\int_\epsilon^t \int_{\mathbb{R}} K_i(t,x;s,y)\, g(s,y)\, dy\, ds,$$

where

$$K_i(t,x;s,y) := \frac{p(t-s,x-y)^2}{p(t,x)^2} \times \int_{\substack{(r_1,\dots,r_M)\in\Delta_M \\ r_i=s}} \int_{\mathbb{R}^{M-1}} \prod_{j\neq i} \Big[\Pi_n(r_j,z_j;t,x)^2 p(r_j,z_j)^{\frac{2}{q-1}}\Big] \prod_{j\neq i} dz_j dr_j. \tag{503}$$

Summing over i, define

$$K(t,x;s,y) := \sum_{i=1}^M K_i(t,x;s,y).$$

Then (501) becomes

$$g(t,x) \le C_M \left(1 + \int_\epsilon^t \int_{\mathbb{R}} K(t,x;s,y)\, g(s,y)\, dy\, ds\right). \tag{504}$$

For $j \neq i$, define

$$A(r_j;t,x) := \int_{\mathbb{R}} \Pi_n(r_j,z_j;t,x)^2\, p(r_j,z_j)^{\frac{2}{q-1}}\, dz_j.$$

Using Remark 3, we are able to provide the following bound

$$\|p(r_j,\cdot)\|_\infty \lesssim r_j^{-1/2}, \tag{505}$$

together with the bridge-density estimate

$$\int_{\mathbb{R}} \Pi_n(r_j,z_j;t,x)^2\, dz_j \lesssim \sqrt{\frac{t}{r_j(t-r_j)}},$$

we obtain

$$A(r_j;t,x) \lesssim r_j^{-1/(q-1)} \sqrt{\frac{t}{r_j(t-r_j)}}.$$

Therefore, returning to (503) we obtain

$$K_i(t,x;s,y) \lesssim \frac{p(t-s,x-y)^2}{p(t,x)^2} \int_{\substack{(r_1,\dots,r_M)\in\Delta_M \\ r_i=s}} \prod_{j\neq i} r_j^{-\frac{1}{q-1}} \left(\frac{t}{r_j(t-r_j)}\right)^{1/2} \prod_{j\neq i} dr_j. \tag{506}$$

We now consider the remaining time integral

$$\int_{\substack{(r_1,\dots,r_M)\in\Delta_M \\ r_i=s}} \prod_{j\neq i} r_j^{-\frac{1}{q-1}} \left(\frac{t}{r_j(t-r_j)}\right)^{1/2} \prod_{j\neq i} dr_j. \tag{507}$$

Rewriting the integrand, we obtain

$$\prod_{j\neq i} r_j^{-\frac{1}{q-1}} \left(\frac{t}{r_j(t-r_j)}\right)^{1/2} = C_t \prod_{j\neq i} r_j^{-\frac{1}{q-1}-\frac{1}{2}} (t-r_j)^{-1/2}, \tag{508}$$

where $C_t > 0$ depends only on t.

Thus, the integral is of Dirichlet type: after the change of variables $u_j = r_j/t$, it becomes an ordered simplex integral of the form

$$\int_{0<u_1<\cdots<u_{M-1}<1} \prod_{j\neq i} u_j^{\alpha-1}(1-u_j)^{\beta-1}\, du_1 \cdots du_{M-1}, \tag{509}$$

with

$$\alpha = 1 - \left(\frac{1}{q-1} + \frac{1}{2}\right), \qquad \beta = \frac{1}{2}.$$

Such integrals are finite whenever the exponents satisfy $\alpha > 0$ and $\beta > 0$. In the present setting, since the simplex is bounded away from 0, the singularity at $r_j = 0$ is avoided. **NOTE:** nearly there, but getting confused about what needs to summable etc, Estimating K allows us to control the iterates of the associated Volterra operator. Since the time integrations are ordered, these iterates are governed by simplex integrals of Dirichlet type, which are summable. Consequently, the Picard iteration for g converges and yields a uniform bound.

E Technical Lemmas

E.1 Mulitdimensional Garsia-Rodemich-Rumsey Inequality

Proposition E.1 (Multidimensinal Garsia–Rodemich–Rumsey Inequality (GRR))**.** *Let $\phi(x)$ and $\varphi(x)$ be positive continuous functions on $(0,\infty)$ such that $\phi(x)$ is convex with $\lim_{x\to\infty}\phi(x)=\infty$, and $\varphi(x)$ is strictly increasing on $(0,\infty)$ with $\varphi(0)=0$. Let f be a continuous function on a subset $R\subset\mathbb{R}^d$. Suppose that*

$$\int_R\int_R \phi\left(\frac{f(x)-f(y)}{\varphi((x-y)/\sqrt{d})}\right) dx\, dy = B < \infty.$$

Then, for all $x, y \in R$,

$$|f(x)-f(y)| \le 8\int_0^{|x-y|} \phi^{-1}\left(\frac{4B}{u^2}\right) d\varphi(u).$$

Proof. See Lemma 2 in [**?**]. ∎

E.2 Youngs Inequality

Lemma E.2 (Young's Inequality)**.** *Let $1 \le p_1, \dots, p_k, r \le \infty$ such that $1/p_1 + \cdots + 1/p_k - (k-1) = 1/r$. If $f_1 \in L^{p_1}(\mathbb{R}^d), \dots, f_k \in L^{p_k}(\mathbb{R}^d)$, then*

$$\|f_1 * \cdots * f_k\|_{L^r} \le \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}.$$

In particular, if $p_j = p$ for all $1 \le j \le k$, then

$$\|f^{*k}\|_{L^r} \le \|f\|_{L^p}^k,$$

provided $k/p - (k-1) = 1/r$.

See [Fol99] for a proof.

E.3 Standard Fourier Analysis Bound

Theorem E.3. *Let $k \ge 2$ be an integer, and set $p = \frac{2k}{2k-1}$. then,*

$$\|f\|_{L^{2k}} \lesssim \|\widehat{f}\|_{L^p}.$$

Proof. Note that $\|f\|_{L^{2k}}^{2k} = \|f^k\|_{L^2}^2 = \|\widehat{f^k}\|_{L^2}^2$, where the last equality follows from Plancherel's theorem. Since the Fourier transform converts pointwise products into convolutions , we have $\widehat{f^k} = (\widehat{f})^{*k}$, and hence $\|f\|_{L^{2k}}^{2k} \lesssim \|(\widehat{f})^{*k}\|_{L^2}^2$. Now apply Young's convolution inequality $k-1$ times (see (E.2)) with coefficients satisfying $\frac{k}{p} - (k-1) = \frac{1}{2}$. Solving for p gives $p = \frac{2k}{2k-1}$. Then $\|(\widehat{f})^{*k}\|_{L^2} \le \|\widehat{f}\|_{L^p}^k$. We can conclude that

$$\|f\|_{L^{2k}} \lesssim \|\widehat{f}\|_{L^{\frac{2k}{2k-1}}}.$$

Interpolating this estimate with Plancherel's theorem $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$, via the Riesz–Thorin interpolation theorem, we obtain that for every $0 \le \theta \le 1$, $\|f\|_{L^{q_\theta}} \lesssim \|\widehat{f}\|_{L^{p_\theta}}$, where $\frac{1}{p_\theta} = \frac{1-\theta}{2} + \theta\frac{2k-1}{2k}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{2} + \theta\frac{1}{2k}$. □

E.4 Dirichlet-Type Integrals

Recall that the Dirichlet density on the simplex is given by

$$f(x_1, \dots, x_k; \alpha_1, \dots, \alpha_k) = \frac{1}{B(\alpha_1, \dots, \alpha_k)} \prod_{i=1}^{k} x_i^{\alpha_i - 1}, \tag{510}$$

for $x_i \in [0,1]$, $1 \le i \le k$, subject to

$$\sum_{i=1}^{k} x_i = 1. \tag{511}$$

Here,

$$B(\alpha_1,\dots,\alpha_k) := \frac{\prod_{j=1}^k \Gamma(\alpha_j)}{\Gamma\Big(\sum_{j=1}^k \alpha_j\Big)} \tag{512}$$

denotes the multivariate Beta function. We will frequently encounter the special case $\alpha_1 = \cdots = \alpha_k = \frac{1}{2}$, for which (512) simplifies to

$$B\left(\frac{1}{2},\dots,\frac{1}{2}\right) = \frac{\pi^{k/2}}{\Gamma(k/2)}. \tag{513}$$

Applying Stirling's formula,

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z}, \qquad z \to \infty, \tag{514}$$

and setting $z = k/2$, yields

$$\Gamma(k/2) \sim \sqrt{2\pi}\left(\frac{k}{2}\right)^{\frac{k}{2}-\frac{1}{2}} e^{-k/2}. \tag{515}$$

Substituting this into (513), we obtain

$$\begin{aligned} B\left(\frac{1}{2},\dots,\frac{1}{2}\right) &= \frac{\pi^{k/2}}{\Gamma(k/2)} \\ &\sim \frac{\pi^{k/2}}{\sqrt{2\pi}\left(\frac{k}{2}\right)^{\frac{k}{2}-\frac{1}{2}} e^{-k/2}} \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{2\pi e}{k}\right)^{k/2}\left(\frac{k}{2}\right)^{1/2}, \qquad k \to \infty. \end{aligned} \tag{516}$$

In particular, the multivariate Beta function decays super-exponentially in k. We now consider integrals over the k-simplex

$$\Delta_k(t) := \{(s_1,\dots,s_k) : 0 < s_1 < \cdots < s_k < t\}. \tag{517}$$

Suppose we wish to estimate an integral of the form

$$I_k(t) := \int_{\Delta_k(t)} (t-s_k)^{\alpha_{k+1}-1} \prod_{j=1}^k (s_j - s_{j-1})^{\alpha_j - 1}\, ds_1 \cdots ds_k, \tag{518}$$

where we adopt the convention $s_0 = 0$, and where

$$\alpha_1,\dots,\alpha_{k+1} > 0. \tag{519}$$

Define the change of variables

$$r_1 = s_1, \qquad r_j = s_j - s_{j-1}, \quad 2 \le j \le k, \qquad r_{k+1} = t - s_k. \tag{520}$$

Then each $r_j > 0$, and

$$r_1 + \cdots + r_{k+1} = t. \tag{521}$$

Further the inverse transformation is

$$s_j = r_1 + \cdots + r_j, \qquad 1 \le j \le k. \tag{522}$$

The Jacobian of the map $(r_1, \ldots, r_k) \mapsto (s_1, \ldots, s_k)$ can be seen to be triangular with determinant 1, and hence

$$ds_1 \cdots ds_k = dr_1 \cdots dr_k. \tag{523}$$

Therefore,

$$I_k(t) = \int_{\substack{r_1+\cdots+r_{k+1}=t \\ r_j>0}} \prod_{j=1}^{k+1} r_j^{\alpha_j - 1}\, dr_1 \cdots dr_k. \tag{524}$$

To evaluate this integral, we can rescale it onto the unit simplex by another change of varible

$$r_i = tu_i, \qquad 1 \le i \le k+1. \tag{525}$$

Since $r_1 + \cdots + r_{k+1} = t$, we obtain that $u_1 + \cdots + u_{k+1} = 1$. and hence the simplex $\{r_i > 0 : r_1 + \cdots + r_{k+1} = t\}$ can be mapped to the unit simplex

$$\Delta_k = \{u_i > 0 : u_1 + \cdots + u_{k+1} = 1\}. \tag{526}$$

It follows from the mapping that

$$r_i^{\alpha_i - 1} = (tu_i)^{\alpha_i - 1} = t^{\alpha_i - 1} u_i^{\alpha_i - 1}, \tag{527}$$

and therefore

$$\prod_{i=1}^{k+1} r_i^{\alpha_i - 1} = t^{\sum_{i=1}^{k+1} (\alpha_i - 1)} \prod_{i=1}^{k+1} u_i^{\alpha_i - 1}. \tag{528}$$

Since there are k free variables,

$$dr_1 \cdots dr_k = t^k\, du_1 \cdots du_k. \tag{529}$$

Combining these powers yields

$$\begin{aligned} I_k &= t^{\sum_{i=1}^{k+1} (\alpha_i - 1)} t^k \int_{\Delta_k} \prod_{i=1}^{k+1} u_i^{\alpha_i - 1}\, du_1 \cdots du_k \\ &= t^{\sum_{i=1}^{k+1} \alpha_i - 1} \int_{\Delta_k} \prod_{i=1}^{k+1} u_i^{\alpha_i - 1}\, du_1 \cdots du_k. \end{aligned} \tag{530}$$

The remaining integral is the classical Dirichlet integral:

$$\int_{\Delta} \prod_{i=1}^{k+1} u_i^{\alpha_i - 1} \, du_1 \cdots du_k = \frac{\prod_{i=1}^{k+1} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k+1} \alpha_i\right)}. \tag{531}$$

Consequently,

$$I_k(t) = t^{\sum_{i=1}^{k+1} \alpha_i - 1} \frac{\prod_{i=1}^{k+1} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k+1} \alpha_i\right)}. \tag{532}$$

If every singularity is of order 1/2, then

$$\alpha_1 = \cdots = \alpha_{k+1} = \frac{1}{2}, \tag{533}$$

then

$$I_k(t) = t^{(k+1)/2-1} \frac{\Gamma(1/2)^{k+1}}{\Gamma((k+1)/2)}. \tag{534}$$

Since $\Gamma(1/2) = \sqrt{\pi}$, this simplifies to

$$I_k(t) = t^{(k-1)/2} \frac{\pi^{(k+1)/2}}{\Gamma((k+1)/2)}. \tag{535}$$

F Notation

$\mathbb{Z}^d$	d-dimensional integer lattice
S_n	Simple random walk at time n
$\mathcal{F}_n$	σ-algebra generated by $(S_0, \ldots, S_n)$
P^{SRW}	Law of the simple random walk
P_h^{TILT}	Tilted path measure with parameter h
$\hat{\phi}^{\mathrm{TILT}}$	Characteristic function under the measure $P_{h_n}^{\mathrm{TILT}}$.
h	Tilt parameter
$\gamma(h)$	Normalizing constant $P^{\mathrm{SRW}}[e^{h(S_1-S_0)}]$
$\rho_h^{\mathrm{TILT}}(t,x)$	(28)
D_k^n	(83)
$\hat{f}$	Fourier transform of a function
$S_k^n(g)$ (93)	

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